

# CONFIGURATIONS OF POINTS AND STRINGS

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## Abstract

Let  $X$  be a smooth projective variety of dimension  $n \geq 2$ . It is shown that a finite configuration of points on  $X$  subject to certain geometric conditions possesses rich inner structure. On the mathematical level this inner structure is a variation of Hodge-like structure. As a consequence one can attach to such point configurations:

- (i) Lie algebras and their representations
- (ii) Fano toric variety whose hyperplane sections are Calabi-Yau varieties.

These features imply that the points cease to be 0-dimensional objects and acquire dynamics of linear operators “propagating” along the paths of a particular trivalent graph. Furthermore following particular linear operators along the “shortest” paths of the graph one creates, for every point of the configuration, a distinguished hyperplane section of the Fano variety in (ii), i.e. the points “open up” to become Calabi-Yau varieties. Thus one is led to a picture which is very suggestive of quantum gravity according to string theory.

## § 0 Introduction

In the last two decades certain domains of mathematics, such as algebraic geometry, have changed dramatically under the influence of ideas originating in physics and string theory in particular. Recasting these ideas as rigorous mathematical theories has been a very fruitful challenge for mathematicians. In return, mathematical thinking deepens and refines physicists’ understanding of string theory.

This paper in a certain sense reverses the above logic. Namely, we will discuss some algebro-geometric results and constructions and suggest that they might have a natural string theoretic interpretation. A word of warning: the author is a total neophyte in physics so we apologize to the expert reader if our “natural” will be more a synonym of naive.

The results we will be exposing concern 0-dimensional subschemes or clusters (according to the terminology suggested by Miles Reid about 20 years ago) on a smooth complex projective variety of complex dimension  $n \geq 2$ . Our recent work, [R1], motivated by purely mathematical considerations, gives rise to a very natural passage from clusters on a smooth projective surface (complex dimension 2) to realm of Lie algebras and their representations, toric varieties whose hyperplane sections are Calabi-Yau varieties (we are interpreting Calabi-Yau variety in a rather broad terms here - it is a connected complex projective variety  $Y$  whose dualizing sheaf  $\omega_Y = \mathcal{O}_Y$ ). The main goal of this paper is to explain the mathematics of this passage in any dimension  $\geq 2$  and to suggest that it might have a string theoretic interpretation.

In the rest of the introduction we will outline the main results and ideas of the paper for

clusters coming from holomorphic vector bundles or locally free sheaves (as it is customary we make no distinction between these notions). This is largely for two reasons. The first one is that these clusters possess technical properties which are necessary for our constructions. Once these technical properties are isolated a passage to a more general setting presents no difficulties. The second reason is that the vector bundles constitute one of the obvious targets for applications of our constructions.

### Outline of the results in a vector bundle setting

Let  $X$  be a smooth complex projective variety of dimension  $n \geq 2$  (all dimensions are over  $\mathbf{C}$ , unless stated otherwise) and let  $\mathcal{E}$  be a holomorphic vector bundle of the same rank  $n$  over  $X$ . Denote by  $\mathcal{O}_X(L)$  the determinant bundle of  $\mathcal{E}$  and assume  $H^i(\mathcal{O}_X(-L)) = 0$ , for all  $i \leq n - 1$ , e.g.  $\mathcal{O}_X(L)$  is an ample line bundle. This is somewhat restrictive assumption but it holds in many geometric situations which motivated our considerations.

In addition, it always will be assumed that  $\mathcal{E}$  has a global section  $e$  whose zero-locus  $Z_e = (e = 0)$  is a set of distinct points, i.e. the section  $e$  has finite number of zeros and every zero of  $e$  has multiplicity 1. These “good” sections of  $\mathcal{E}$  are considered equivalent if they differ by a nonzero scalar multiple. Their equivalence classes form a Zariski open subset  $\overset{\circ}{U}_{\mathcal{E}}$  of the projective space  $\mathbf{P}(H^0(\mathcal{E}))$ . Over every point  $[e] \in \overset{\circ}{U}_{\mathcal{E}}$  we have its zero-locus  $Z_e = (e = 0)$  and as  $[e]$  varies in  $\overset{\circ}{U}_{\mathcal{E}}$  the zero-loci  $Z_e$  will move in  $X$  and form the family

$$p_{\overset{\circ}{U}_{\mathcal{E}}} : \overset{\circ}{Z} \longrightarrow \overset{\circ}{U}_{\mathcal{E}} \quad (0.1)$$

of 0-dimensional subvarieties of  $X$ . Observe that  $p_{\overset{\circ}{U}_{\mathcal{E}}}$  is an unramified covering whose degree is the  $n$ -th Chern number of  $\mathcal{E}$ , i.e.

$$d := \deg(p_{\overset{\circ}{U}_{\mathcal{E}}}) = \int_{[X]} c_n(\mathcal{E}) \quad (0.2)$$

is the integrated  $n$ -th Chern class  $c_n(\mathcal{E})$  of  $\mathcal{E}$ .

The variety  $\overset{\circ}{Z}$  is a natural geometric object associated with  $(X, \mathcal{E})$  and it will be called the *universal configuration* of  $\mathcal{E}$ . The open set  $\overset{\circ}{U}_{\mathcal{E}}$  is the parameter space for these configurations. We suggest that this is only “visible” or “classical” parameters of the point configurations and  $\overset{\circ}{Z}$  itself is only a “visible” part of the “universe” associated to  $\mathcal{E}$ . A construction of the space of “hidden” parameters for point configurations  $Z_e, [e] \in \overset{\circ}{U}_{\mathcal{E}}$ , and uncovering its various features constitute the mathematical content of the paper.

### Part I: The jacobian of $\overset{\circ}{U}_{\mathcal{E}}$ or the space of “hidden” parameters

The main result of this part is a construction of the torsion free sheaf  $\mathbf{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}$  over  $\overset{\circ}{U}_{\mathcal{E}}$  whose fibre  $\mathbf{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}([e])$  at  $[e] \in \overset{\circ}{U}_{\mathcal{E}}$  is isomorphic to the group of extensions  $Ext^{n-1}(\mathcal{I}_{Z_e}(L), \mathcal{O}_X)$ , where  $\mathcal{I}_{Z_e}$  is the sheaf of ideals of  $Z_e$ . In the sequel this group will be denoted by  $Ext_{Z_e}^{n-1}(L)$ .

Let  $\delta(\overset{\circ}{U}_{\mathcal{E}})$  be the rank of  $\mathbf{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}$  and let  $Sing(\mathbf{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1})$  be the singularity set of  $\mathbf{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}$ ,

i.e. the subset of  $\overset{\circ}{U}_{\mathcal{E}}$  where the rank of fibres of  $\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}$  exceeds  $\delta(\overset{\circ}{U}_{\mathcal{E}})$ . From general properties of torsion free sheaves (see e.g. [O-S-S]) it follows that  $\text{Sing}(\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1})$  is a closed proper subscheme of  $\overset{\circ}{U}_{\mathcal{E}}$ . Set  $\overset{\circ}{U}'_{\mathcal{E}} = \overset{\circ}{U}_{\mathcal{E}} \setminus \text{Sing}(\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1})$ . The restriction of  $\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1}$  to  $\overset{\circ}{U}'_{\mathcal{E}}$  is now a locally free sheaf of rank  $\delta(\overset{\circ}{U}_{\mathcal{E}})$ . In particular,  $\dim(Ext_{Z_e}^{n-1}(L)) = \delta(\overset{\circ}{U}_{\mathcal{E}})$ , for every  $[e] \in \overset{\circ}{U}'_{\mathcal{E}}$ .

Define

$$\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}} = \mathbf{P}(\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1} \otimes \mathcal{O}_{\overset{\circ}{U}'_{\mathcal{E}}}) \quad (0.3)$$

to be the projectivized bundle associated to  $\text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1} \otimes \mathcal{O}_{\overset{\circ}{U}'_{\mathcal{E}}}$ . This is our space of “hidden” parameters for the configuration  $\overset{\circ}{Z}$  or what we call *the nonabelian jacobian* of  $\overset{\circ}{U}_{\mathcal{E}}$ .

By definition our jacobian comes with the projection

$$\pi_{\overset{\circ}{U}_{\mathcal{E}}} : \mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}} \longrightarrow \overset{\circ}{U}_{\mathcal{E}} \quad (0.4)$$

and the line bundle  $\mathcal{O}_{\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}}(1)$  defined so that its direct image

$$\pi_{\overset{\circ}{U}_{\mathcal{E}}*} \left( \mathcal{O}_{\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}}(1) \right) = \left( \text{Ext}_{\overset{\circ}{U}_{\mathcal{E}}}^{n-1} \otimes \mathcal{O}_{\overset{\circ}{U}'_{\mathcal{E}}} \right)^*$$

Thus the construction of the space of “hidden” parameters amounts to attaching the projective space  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  to every point  $[e] \in \overset{\circ}{U}_{\mathcal{E}}$  of the space of “visible” parameters of the universal configuration of  $\mathcal{E}$ . It is clear that we get something new only if the rank  $\delta(\overset{\circ}{U}_{\mathcal{E}}) \geq 2$ . This always will be assumed unless said otherwise. Observe that for  $n = 1$  our construction *never* gives anything new. So an existence of nontrivial nonabelian jacobian is a distinguished feature of higher dimensional geometry.

As the space of “visible” parameters  $\overset{\circ}{U}_{\mathcal{E}}$  governs the “classical” motion of our configurations in  $X$ , we expect that the “hidden” parameters in  $\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}$  will reveal some “quantum” properties of the universal configuration  $\overset{\circ}{Z}$ . This is indeed the case and the feature of  $\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}$  responsible for this is a Hodge-like structure and its variation along the fibres of  $\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}$ .

## Part II: A variation of Hodge-like structure associated to $\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}$

We begin by explaining what happens on the fibre of  $\mathbf{J}_{\overset{\circ}{U}_{\mathcal{E}}}$  over a point  $[e] \in \overset{\circ}{U}_{\mathcal{E}}$ . On the “classical” level over  $[e]$  lies the configuration  $Z_e$  of  $d$  distinct points (particles) on  $X$ . The space of functions  $H^0(\mathcal{O}_{Z_e})$  on  $Z_e$  can be viewed as the space of classical observables of  $Z_e$ . Attaching to  $[e]$  the “hidden” parameters  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  has the following effect.

There is a nonempty Zariski open subset  $\check{\mathbf{J}}([e])$  of  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  such that for every  $[\alpha] \in \check{\mathbf{J}}([e])$  the space of classical observables  $H^0(\mathcal{O}_{Z_e})$  admits a distinguished direct sum decomposition

$$H^0(\mathcal{O}_{Z_e}) = \bigoplus_{p=0}^{w(\overset{\circ}{U}_{\mathcal{E}})} \mathbf{H}^p([e], [\alpha]) \quad (0.5)$$

where the number of summands  $w = w(\overset{\circ}{U}_{\mathcal{E}}) \geq 1$  is called the weight of  $\overset{\circ}{U}_{\mathcal{E}}$ .

The summand  $\mathbf{H}^0([e], [\alpha])$  plays a special role.

**Lemma 0.1** 1)  $\mathbf{H}^0([e], [\alpha])$  contains the subspace  $H^0(\mathcal{O}_X) = \mathbf{C}$  of constant functions on  $Z_e$ .

2) There is a natural isomorphism of vector spaces

$$\theta_{[e]}(\alpha) : \mathbf{H}^0([e], [\alpha]) \longrightarrow \text{Ext}_{Z_e}^{n-1}(L) \quad (0.6)$$

which sends the constant function  $1 \in \mathbf{H}^0([e], [\alpha])$  to the extension class  $\alpha \in \text{Ext}_{Z_e}^{n-1}(L)$ .

The essential point is that the decomposition (0.5) varies with  $([e], [\alpha])$ . In particular, we study its variation with respect to the “hidden” parameter  $[\alpha]$ . By virtue of the identification in (0.6) this variation is closely related to the multiplication in the ring  $H^0(\mathcal{O}_{Z_e})$  by elements in  $\mathbf{H}^0([e], [\alpha])$ . It turns out that the multiplication by  $t \in \mathbf{H}^0([e], [\alpha])$  preserves the subspace

$$\tilde{\mathbf{H}}_{-w}([e], [\alpha]) = \bigoplus_{p=0}^{w-1} \mathbf{H}^p([e], [\alpha]) \quad (0.7)$$

and acts on this decomposition by shifting the grading (up or down) at most by 1. In other words, if  $D(t)$  denotes the operator of the multiplication by  $t$  in  $H^0(\mathcal{O}_{Z_e})$  and if  $D_p(t)$  is its restriction to the summand  $\mathbf{H}^p([e], [\alpha])$ , for  $p = 0, \dots, w-1$ , then

$$D_p(t) : \mathbf{H}^p([e], [\alpha]) \longrightarrow \mathbf{H}^{p-1}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha]) \quad (0.8)$$

with an understanding that  $\mathbf{H}^{-1} = 0$ . Denote by  $D_p^-(t)$  (resp.  $D_p^0(t)$  and  $D_p^+(t)$ ) the projection of  $D_p(t)$  on the first (resp. the second and the third) summand in (0.8) with an understanding that  $D_0^-(t) = D_{w-1}^+(t) = 0$ .

Set

$$D^\pm(t) = \sum_{p=0}^{w-1} D_p^\pm(t), \quad D^0(t) = \sum_{p=0}^{w-1} D_p^0(t) \quad (0.9)$$

The operator  $D(t)$  restricted to the subspace  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$  in (0.7) admits the following decomposition

$$D(t) = D^-(t) + D^0(t) + D^+(t) \quad (0.10)$$

where  $D^0(t)$  preserves the grading in (0.7) and  $D^\pm(t)$  shifts it by  $\pm 1$ . Thus the “hidden” variable  $[\alpha]$  produces the direct sum decomposition (0.5) and defines the subspace

$$\{D^\pm(t), D^0(t) \mid t \in \mathbf{H}^0([e], [\alpha])\} \quad (0.11)$$

of  $\text{End}(\tilde{\mathbf{H}}_{-w}([e], [\alpha]))$ . In other words, *a priori* static finite set of points is turned into a dynamic object. The dynamics manifests itself in a variation of the decomposition (0.7) with respect to the “hidden” variable  $[\alpha]$ . The variation itself is governed by the decomposition (0.10). In particular, we see that our classical observable, a function  $t \in \mathbf{H}^0([e], [\alpha])$ , acquires now the status of a system of linear operators and while the multiplication operators  $D(t)$  are obviously commuting the Lie subalgebra of  $\text{End}(\tilde{\mathbf{H}}_{-w}([e], [\alpha]))$  generated by the operators in (0.11), in general, is not abelian. All this indicates that our “hidden” parameter  $[\alpha]$  is capable

of detecting “quantum” properties of the configuration  $Z_e$ .

Until now we have been discussing the situation on the fibre of  $\mathbf{J}_{\overset{\circ}{U}_\varepsilon}$  over a point  $[e]$  in  $\overset{\circ}{U}_\varepsilon'$ . What happens when we vary  $[e]$ ? First of all the rings  $H^0(\mathcal{O}_{Z_e})$  fit together to form the locally free sheaf  $p_{U_\varepsilon^*} \mathcal{O}_{\overset{\circ}{Z}}$ , the direct image of  $\mathcal{O}_{\overset{\circ}{Z}}$  under the morphism  $p_{U_\varepsilon}$  in (0.1). So to generalize our results we need to consider the pullback  $\pi_{U_\varepsilon}^*(p_{U_\varepsilon^*} \mathcal{O}_{\overset{\circ}{Z}})$  of  $p_{U_\varepsilon^*} \mathcal{O}_{\overset{\circ}{Z}}$  via the projection  $\pi_{U_\varepsilon}$  in (0.4). We can now state the sheaf version of our discussion.

**Theorem 0.2** 1) There is a distinguished nonempty Zariski open subset  $\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}$  of  $\mathbf{J}_{\overset{\circ}{U}_\varepsilon}$  and an integer  $w(\overset{\circ}{U}_\varepsilon) \geq 1$  such that the sheaf  $\pi_{U_\varepsilon}^*(p_{U_\varepsilon^*} \mathcal{O}_{\overset{\circ}{Z}}) \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}}$  admits a distinguished direct sum decomposition

$$\pi_{U_\varepsilon}^*(p_{U_\varepsilon^*} \mathcal{O}_{\overset{\circ}{Z}}) \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}} = \bigoplus_{p=0}^{w(\overset{\circ}{U}_\varepsilon)} \mathbf{H}_{U_\varepsilon}^p \quad (0.12)$$

The integer  $w = w(\overset{\circ}{U}_\varepsilon)$  is called the weight of  $\overset{\circ}{U}_\varepsilon$  (if no ambiguity is likely we will simplify the above notation by omitting the reference to  $\overset{\circ}{U}_\varepsilon$ ).

2) The summand  $\mathbf{H}^0$  in (0.12) has the following properties.

- (a)  $\mathcal{O}_{\check{\mathbf{J}}}$  is a subbundle of  $\mathbf{H}^0$ .
- (b) There is a natural isomorphism

$$\theta : \mathbf{H}^0 \otimes \mathcal{O}_{\check{\mathbf{J}}}(-1) \longrightarrow \pi_{U_\varepsilon}^* \text{Ext}_{U_\varepsilon}^{n-1} \quad (0.13)$$

which induces the following commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathcal{O}_{\check{\mathbf{J}}}(-1) & \xlongequal{\quad} & \mathcal{O}_{\check{\mathbf{J}}}(-1) \\ \downarrow & & \downarrow \\ \mathbf{H}^0 \otimes \mathcal{O}_{\check{\mathbf{J}}}(-1) & \xrightarrow{\theta} & \pi_{U_\varepsilon}^* \text{Ext}_{U_\varepsilon}^{n-1} \\ \downarrow & & \downarrow \\ (\mathbf{H}^0 / \mathcal{O}_{\check{\mathbf{J}}}) \otimes \mathcal{O}_{\check{\mathbf{J}}}(-1) & \xrightarrow{\theta'} & \mathcal{T}_{\check{\mathbf{J}}/\overset{\circ}{U}_\varepsilon}(-1) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array} \quad (0.14)$$

where  $\mathcal{T}_{\check{\mathbf{J}}/\overset{\circ}{U}_\varepsilon}$  is the relative tangent bundle of  $\pi_{U_\varepsilon} : \check{\mathbf{J}} \longrightarrow \overset{\circ}{U}_\varepsilon$  and the column on the right is the relative Euler sequence on  $\check{\mathbf{J}}$  tensored with  $\mathcal{O}_{\check{\mathbf{J}}}(-1)$ .

Next we state the sheaf version of the endomorphisms in (0.11).

**Theorem 0.3** Set

$$\tilde{\mathbf{H}}_{-w} = \bigoplus_{p=0}^{w-1} \mathbf{H}^p \quad (0.15)$$

1) The multiplication in the sheaf of rings  $\pi_{\overset{\circ}{U}_{\mathcal{E}}}^*(p_{\overset{\circ}{U}_{\mathcal{E}}*}\mathcal{O}_{\overset{\circ}{Z}})$  gives rise to the following morphism of sheaves

$$D : \tilde{\mathbf{H}}_{-w} \longrightarrow (\mathbf{H}^0)^* \otimes \tilde{\mathbf{H}}_{-w} \quad (0.16)$$

which admits the decomposition

$$D = D^- + D^0 + D^+ \quad (0.17)$$

where  $D^0, D^\pm \in \text{Hom}(\tilde{\mathbf{H}}_{-w}, (\mathbf{H}^0)^* \otimes \tilde{\mathbf{H}}_{-w})$

2) The morphism  $D^0$  (resp.  $D^\pm$ ) is a morphism of degree 0 (resp.  $\pm 1$ ) with respect to the grading in (0.15). Furthermore the morphisms  $D^0, D^\pm$  are subject to the following relations

$$(i) \quad D^\pm \wedge D^\pm = 0$$

$$(ii) \quad D^0 \wedge D^\pm + D^\pm \wedge D^0 = 0 \quad (0.18)$$

$$(iii) \quad D^0 \wedge D^0 + D^+ \wedge D^- + D^- \wedge D^+ = 0$$

### Part III: A geometric interpretation of Theorem 0.3

The commutativity of the multiplication in  $\pi_{\overset{\circ}{U}_{\mathcal{E}}}^*(p_{\overset{\circ}{U}_{\mathcal{E}}*}\mathcal{O}_{\overset{\circ}{Z}})$  translates into the fact that  $D^2 = D \wedge D = 0$ , i.e.  $D$  is a Higgs morphism of  $\tilde{\mathbf{H}}_{-w}$  with values in  $(\mathbf{H}^0)^*$  (see [S] or Definition 3.2,[R1], for definitions). To extract geometry from Theorem 0.3 we deform the morphism  $D$  using the direct sum decomposition (0.15). This gives us a particular family of Higgs morphisms parametrized by an affine variety explicitly defined by a system of quadratic equations (Lemma 5.1). This variety is denoted by  $\hat{H}_{\overset{\circ}{U}_{\mathcal{E}}}$ . Passing to the homothety equivalence classes of Higgs morphisms parametrized by  $\hat{H}_{\overset{\circ}{U}_{\mathcal{E}}}$  we obtain the quotient

$$H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}} = \hat{H}_{\overset{\circ}{U}_{\mathcal{E}}} / \mathbf{C}^* \quad (0.19)$$

which is called the nonabelian Albanese of  $\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}$ .

**Theorem 0.4** Let the weight  $w = w(\overset{\circ}{U}_{\mathcal{E}}) \geq 2$ . Then  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  is a projective toric variety of dimension  $w - 1$ . The variety comes equipped with a very ample invertible sheaf  $\mathcal{O}_{H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}}(1)$  such that the dualizing sheaf  $\omega_{H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}} = \mathcal{O}_{H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}}(-1)$ , i.e. the variety  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  is a (toric) Fano variety. In particular, if the weight  $w \geq 3$ , then the hyperplane sections of  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  are Calabi-Yau varieties of dimension  $w - 2$ .

The nonabelian Albanese of  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  depends only on the weight  $w$  and the relations between the morphisms  $D^0, D^\pm$  in (0.18). So it completely “forgets” about the variety  $X$  and the universal configuration  $\overset{\circ}{Z}$ . However, there are various ways to remedy this by constructing correspondences between  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  and  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$ . In particular, there is a geometric correspondence described as follows.

**Theorem 0.5** *There is a natural geometric correspondence between  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  and  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$  given by the morphism*

$$CY_{\overset{\circ}{U}_{\mathcal{E}}} : \check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}} \longrightarrow \mathbf{P}(H^0(\mathcal{O}_{H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}}(d))) \quad (0.20)$$

where  $d$  is the  $n$ -th Chern number of  $\mathcal{E}$ . This morphism sends a point  $([e], [\alpha])$  of  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  to a cycle of the hyperplane sections of  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$ . More precisely, the (projectivized) section  $CY_{\overset{\circ}{U}_{\mathcal{E}}}([e], [\alpha])$  corresponds to the divisor

$$\sum_{z \in Z_e} H_{z, [\alpha]} \quad (0.21)$$

where  $H_{z, [\alpha]}$  is a hyperplane section of  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$  intrinsically associated to a point  $(z, [\alpha]) \in (Z_e, [\alpha])$ .

If  $w \geq 3$ , then every  $H_{z, [\alpha]}$  is a Calabi-Yau variety and the morphism  $CY_{\overset{\circ}{U}_{\mathcal{E}}}$  associates to every point of  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  a cycle of Calabi-Yau varieties. For this reason  $CY_{\overset{\circ}{U}_{\mathcal{E}}}$  is called Calabi-Yau cycle map.

The result of Theorem 0.5 tells us that once we fix  $[\alpha]$  in the fibre of  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  over a point  $[e]$  in  $\overset{\circ}{U}'_{\mathcal{E}}$ , every point  $z \in Z_e$  “opens up” and reveals the Calabi-Yau variety  $H_{z, [\alpha]}$ . Thus our constructions show that over every point of the “visible universe”  $\overset{\circ}{Z}$  “seats” a Calabi-Yau variety of dimension  $w-2$ . These Calabi-Yau varieties have moduli: for every “classical” configuration  $Z_e$  this moduli is our “hidden” parameter  $[\alpha]$  moving in the open subset  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}} \cap \mathbf{P}(Ext_{Z_e}^{n-1}(L))$ . Furthermore the hyperplane sections  $H_{z, [\alpha]}$  in (0.21) arise naturally as a kind of propagation of the  $\delta$ -function  $\delta_z$  supported at  $z \in Z_e$ . This propagation is provided by the morphisms  $D^0, D^\pm$  in Theorem 0.3. The main point here is that the data of the decomposition (0.15) and the morphisms  $D^0, D^\pm$  can be organized in a trivalent graph which is denoted  $G(\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}})$ . So the above mentioned propagation is obtained by following the paths of the graph  $G(\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}})$  with endomorphisms from the space (0.11). This way one obtains “strings”=strings of operators intrinsically attached to every point  $(z, [\alpha]) \in (Z_e, [\alpha])$ . In particular, the creation of  $H_{z, [\alpha]}$  could be regarded as a “vibration” of a string of operators according to the pattern of the graph  $G(\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}})$ .

Thus what emerges from our results is a picture of “universe” = “visible”+“hidden” associated to the universal configuration  $\overset{\circ}{Z}$  which is very suggestive of the one predicted by string theory. From this perspective our results could be succinctly formulated as follows.

**Meta-principle** *All smooth complex projective varieties of dimension  $\geq 2$  have a string theoretic nature:*

- *points on these varieties have “hidden” parameters*
- *in the space of these “hidden” parameters points are no longer 0-dimensional but become “strings” in a precise mathematical sense*
- *these “strings” “vibrate” according to the pattern of a certain trivalent graph and this vibration creates Calabi-Yau varieties.*

In the main body of the paper we present proofs either complete or their sketch referring for details to [R1]. This reference contains most of the ideas exposed here. The only really new result of this paper is a generalization of the results in [R1] to an arbitrary dimension  $\geq 2$ .

The rest of the paper is organized as follows.

- In §1 definitions of the universal cluster and the universal configuration are given.
- In §2 we define the space of “hidden” parameters of the universal cluster.
- In §3 the Hodge-like structure alluded to in **Part II** is defined.
- In §4 we discuss mathematics of theorems stated in **Part II** as well as speculate on their string theoretic interpretation:
  - §§4.1,4.2 contain proofs of Theorem 0.2 and Theorem 0.3 respectively,
  - §4.3 treats a trivalent graph which is an explicit mechanism for transforming classical observables=functions to “quantum” observables=path-operators,
  - §4.4 is devoted to speculations about possible string theoretic nature of our constructions.
- **Part III** is treated in §5.
- In §6 we spell out conditions on a family of clusters for which all our constructions remain valid. This allows to generalize all our results to families of clusters which *do not* necessarily come from vector bundles. This becomes important in dimension  $\geq 3$ , where one no longer has an equivalence between extension classes and torsion free sheaves (as it is the case in dimension 2).
- In §7 we illustrate our general considerations by an example of complete intersection clusters.

It is a pleasure to thank Vladimir Roubtsov whose encouragement to present our mathematical results for physicists was instrumental.

## § 1 The universal configuration or the “visible universe” associated to $\mathcal{E}$

Let  $\mathcal{E}$  be a holomorphic vector bundle of rank  $n$  over an  $n$ -dimensional ( $n \geq 2$ ) smooth complex projective variety  $X$ . The basic topological invariants of  $\mathcal{E}$  are its Chern cohomology classes  $c_i(\mathcal{E}) \in H^{2i}(X, \mathbf{Z}), i = 0, \dots, n$ . Under some favorable circumstances the Poincaré duals of  $c_i(\mathcal{E})$  can be represented by algebraic subvarieties or, more generally, by subschemes of  $X$ . We will be particularly interested in the top Chern class  $c_n(\mathcal{E})$ . It is well-known (see e.g. [G-H]) that if  $\mathcal{E}$  has a global section, say  $e$ , such that its zero-locus  $Z_e = (e = 0)$  is 0-dimensional, then  $Z_e$  is a representative of the Poincaré dual of  $c_n(\mathcal{E})$ . In particular, the degree  $d = \deg(Z_e)$ , i.e. the number of zeros of  $e$  counted with their multiplicities, is given by

$$d = \deg(Z_e) = \int_{[X]} c_n(\mathcal{E}) \quad (1.1)$$

the Gauss-Bonnet type formula.

One can view  $Z_e$  as a geometric realization of  $c_n(\mathcal{E})$ . From now on we assume that the vector bundle  $\mathcal{E}$  is subject to the following condition

$$H^0(\mathcal{E}) \neq 0 \text{ and for some } e \in H^0(\mathcal{E}) \text{ the zero-locus } Z_e = (e = 0) \text{ is 0-dimensional} \quad (1.2)$$

This implies in particular that  $d = \deg(Z_e) > 0$ .

The projective space  $\mathbf{P}(H^0(\mathcal{E}))$  parametrizes the zero-loci  $Z_e$  as  $e$  moves in  $H^0(\mathcal{E}) \setminus \{0\}$ . We distinguish in  $\mathbf{P}(H^0(\mathcal{E}))$  the subset  $U_{\mathcal{E}}$  corresponding to the sections of  $\mathcal{E}$  with 0-dimensional zero-loci. This is a Zariski open subset of  $\mathbf{P}(H^0(\mathcal{E}))$ . The assumption (1.2) implies that  $U_{\mathcal{E}}$  is nonempty.

Consider the incidence correspondence

$$X \times U_{\mathcal{E}} \supset \mathcal{Z} = \{(x, [e]) \in X \times U_{\mathcal{E}} \mid e(x) = 0\} \quad (1.3)$$

From the definition of  $U_{\mathcal{E}}$  it follows that the projection

$$p_2 : \mathcal{Z} \longrightarrow U_{\mathcal{E}} \quad (1.4)$$

on the second factor of  $X \times U_{\mathcal{E}}$  is a finite morphism of degree  $d$ . Thus  $\mathcal{Z}$  can be characterized as the largest family of 0-dimensional subschemes of  $X$  which provide geometric realization of  $c_n(\mathcal{E})$ . We will call it *the universal cluster* of  $\mathcal{E}$  or the geometric  $n$ -th Chern class of  $\mathcal{E}$ .

The branch locus of the finite morphism  $p_2$  in (1.4) corresponds to sections of  $\mathcal{E}$  with some of its zeros being multiple.

**Definition 1.1** A global section  $e$  of  $\mathcal{E}$  is called regular if its zero-locus  $Z_e$  is a finite set and every zero of  $e$  has multiplicity 1.

From now on we assume that  $\mathcal{E}$  has a regular section and we denote by  $\overset{\circ}{U}_{\mathcal{E}}$  the subset of  $U_{\mathcal{E}}$  corresponding to the regular sections of  $\mathcal{E}$ , i.e.  $\overset{\circ}{U}_{\mathcal{E}}$  is the set of regular “values” of  $p_2$ . Denote by  $\overset{\circ}{\mathcal{Z}}$  the inverse image of  $\overset{\circ}{U}_{\mathcal{E}}$  with respect to the morphism  $p_2$  in (1.4). Then its restriction

$$p_{\overset{\circ}{U}_{\mathcal{E}}} : \overset{\circ}{\mathcal{Z}} \longrightarrow \overset{\circ}{U}_{\mathcal{E}} \quad (1.5)$$

is an unramified covering of degree  $d$ .

**Definition 1.2** The variety  $\overset{\circ}{\mathcal{Z}}$  is called the universal configuration associated to  $\mathcal{E}$  (or simply the universal configuration of  $\mathcal{E}$ ).

**Remark 1.3** The fact that  $\overset{\circ}{U}_{\mathcal{E}}$  is nonempty is certainly a restrictive assumption. It holds, for example, for vector bundles generated by global sections. Such bundles arise naturally in geometric situations, e.g. from a (non-constant) morphism of  $X$  into the Grassmannian  $Gr(n, V)$  of  $n$ -planes of a given complex vector space  $V$ . So vector bundles with regular sections should be viewed as “geometric” bundles.

The variety  $\overset{\circ}{\mathcal{Z}}$  could be thought of as a configuration of points (particles) moving in  $X$ . The open set  $\overset{\circ}{U}_{\mathcal{E}}$  is the space of “visible” or “classical” parameters of this configuration. We now turn to a construction of the space of its “hidden” parameters.

## § 2 The sheaf of extensions $\text{Ext}_{U_{\mathcal{E}}}^{n-1}$ and the space of “hidden” parameters associated to $\mathcal{Z}$

In this section the construction outlined in **Part I** of the introduction will be given. To avoid unnecessary technical complications it will be assumed that the determinant bundle  $\mathcal{O}_X(L)$  of  $\mathcal{E}$  is subject to the following

$$H^i(\mathcal{O}_X(-L)) = 0, \quad \forall i \leq n-1 \quad (2.1)$$

This condition holds, for example, if  $\mathcal{O}_X(L)$  is ample.

There are some simple algebro-geometric reasons for considering the extension groups

$$\text{Ext}_{Z_e}^{n-1}(L) = \text{Ext}^{n-1}(\mathcal{I}_{Z_e}(L), \mathcal{O}_X) \quad (2.2)$$

where  $[e] \in U_{\mathcal{E}}$  and  $\mathcal{I}_{Z_e}$  is the ideal sheaf of  $Z_e$ . Let us discuss them briefly.

It is well-known fact that  $Z_e$  fails to impose independent conditions on the linear system  $|L + K_X|$ , where  $K_X$  is the canonical divisor of  $X$ . For points to impose (or not) independent conditions on a linear system is a classical algebro-geometric notion. To put it in modern terms in our context consider the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_Z(L + K_X) \longrightarrow \mathcal{O}_X(L + K_X) \xrightarrow{\rho_Z} \mathcal{O}_Z(L + K_X) \longrightarrow 0 \quad (2.3)$$

where  $Z$  is a 0-dimensional subscheme of  $X$  and  $\mathcal{I}_Z$  is its sheaf of ideals.

**Definition 2.1** (i) A cluster  $Z$  is said to fail to impose independent conditions on the linear system  $|L + K_X|$  or, equivalently and more briefly,  $Z$  is called  $L$ -special, iff the homomorphism

$$\rho_Z : H^0(\mathcal{O}_X(L + K_X)) \longrightarrow H^0(\mathcal{O}_Z(L + K_X))$$

induced from the long cohomology sequence of (2.3) fails to be surjective.

(ii) The number  $\delta(L, Z) = \deg Z - \text{rk } \rho_Z$ , where  $\text{rk } \rho_Z$  stands for the rank of  $\rho_Z$ , is called the index of  $L$ -speciality of  $Z$ .

If we assume that the line bundle  $\mathcal{O}_X(L + K_X)$  is generated by global sections and hence defines the morphism

$$\phi_{L+K_X} : X \longrightarrow \mathbf{P}(H^0(\mathcal{O}_X(L + K_X))^*)$$

then the index of speciality tells us that the image of  $Z$  under  $\phi_{L+K_X}$  spans a projective subspace of dimension  $\deg Z - \delta(L, Z) - 1$ . So, for example, to say that 2 points fail to impose independent conditions on the linear system  $|L + K_X|$  or, equivalently, 2 points are  $L$ -special, means that these points are mapped by  $\phi_{L+K_X}$  onto the same point,

3 points are  $L$ -special means that the images of these points are on a line,

4 points are  $L$ -special means that the images of these points are on a plane, etc..

Thus it should be clear that the notion of  $L$ -speciality is a useful concept for studying special projective properties of configuration of points on  $X$  with respect to the so called *adjoint* linear system  $|L + K_X|$ . If we venture into the physical analogy and think of our point configuration as particles moving in  $X$ , then  $L$ -speciality of a configuration might be regarded as a linear constraint on its motion.

We now take  $Z = Z_e$  and examine how the notion of its  $L$ -speciality is related to the extension group  $\text{Ext}_{Z_e}^{n-1}(L)$  in (2.2). This relation becomes clear if one unravels the long cohomology sequence of (2.3) beyond the  $H^0$ -terms. Namely, one obtains

$$\text{coker}(\rho_{Z_e}) = H^1(\mathcal{I}_{Z_e}(L + K_X))$$

where we also used our vanishing hypothesis in (2.1). Applying Serre duality yields

$$H^1(\mathcal{I}_{Z_e}(L + K_X))^* = \text{Ext}^{n-1}(\mathcal{I}_{Z_e}(L + K_X), \mathcal{O}_X(K_X)) = \text{Ext}^{n-1}(\mathcal{I}_{Z_e}(L), \mathcal{O}_X) \quad (2.4)$$

Of course the above argument holds for any subscheme of  $X$ . A particular feature of  $Z_e$  which insures that it is  $L$ -special is the fact that  $Z_e$  comes with a nonzero extension class in  $\text{Ext}_{Z_e}^{n-1}(L)$ . This extension class is defined by the Koszul sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^n \mathcal{E}^* & \xrightarrow{e} & \cdots & \xrightarrow{e} & \wedge^2 \mathcal{E}^* & \xrightarrow{e} & \mathcal{E}^* \xrightarrow{e} \mathcal{I}_{Z_e} \longrightarrow 0 \\ & & \parallel & & & & & & \\ & & \mathcal{O}_X(-L) & & & & & & \end{array} \quad (2.5)$$

where  $\mathcal{E}^*$  is the dual of  $\mathcal{E}$  and the morphisms are contractions with  $e$ , a section defining  $Z_e$ . This sequence is exact precisely when  $Z_e$  is 0-dimensional. So (2.5) determines a particular extension class

$$\alpha_e \in \text{Ext}^{n-1}(\mathcal{I}_{Z_e}, \mathcal{O}_X(-L)) = \text{Ext}^{n-1}(\mathcal{I}_{Z_e}(L), \mathcal{O}_X) = \text{Ext}_{Z_e}^{n-1}(L)$$

which is nonzero because  $\mathcal{E}$  is locally free.

Thus what comes out of the above discussion is that to every  $[e]$  in  $U_{\mathcal{E}}$  we can associate a nonzero complex vector space  $\text{Ext}_{Z_e}^{n-1}(L)$ . These vector spaces fit together to form a sheaf over  $U_{\mathcal{E}}$ . This is the content of the following.

**Proposition 2.2** *There exists a torsion free sheaf  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  over  $U_{\mathcal{E}}$  such that its fibre  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}([e])$  is isomorphic to the extension group  $\text{Ext}_{Z_e}^{n-1}(L)$  in (2.2), for every  $[e] \in U_{\mathcal{E}}$ .*

*Proof.* Let  $\mathcal{I}_{\mathcal{Z}}$  be the ideal sheaf of the universal cluster  $\mathcal{Z} \subset X \times U_{\mathcal{E}}$  defined in (1.3) and consider the defining sequence of  $\mathcal{Z}$

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{X \times U_{\mathcal{E}}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0 \quad (2.6)$$

Denote by  $p_i$  the projection of  $X \times U_{\mathcal{E}}$  onto the  $i$ -th factor,  $i = 1, 2$ . Taking the pullback  $p_1^* \mathcal{O}_X(L + K_X)$  and tensoring (2.6) with it we obtain

$$0 \longrightarrow \mathcal{I}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X) \longrightarrow p_1^* \mathcal{O}_X(L + K_X) \longrightarrow \mathcal{O}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X) \longrightarrow 0 \quad (2.7)$$

Taking the direct image with respect to  $p_2$  yields

$$H^0(L + K_X) \otimes \mathcal{O}_{U_{\mathcal{E}}} \longrightarrow p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)) \longrightarrow R^1 p_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)) \longrightarrow 0 \quad (2.8)$$

Dualizing this sequence gives the following

$$0 \longrightarrow (R^1 p_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)))^* \longrightarrow (p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)))^* \longrightarrow H^0(L + K_X)^* \otimes \mathcal{O}_{U_{\mathcal{E}}} \quad (2.9)$$

Define

$$\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1} := (R^1 p_{2*}(\mathcal{I}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)))^* \quad (2.10)$$

From (2.9) it follows that  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  is a nonzero subsheaf of the locally free sheaf  $(p_{2*}(\mathcal{O}_{\mathcal{Z}} \otimes p_1^* \mathcal{O}_X(L + K_X)))^*$ . Hence  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  is torsion free. Furthermore by definition the fibre of  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  at  $[e] \in U_{\mathcal{E}}$  is  $H^1(\mathcal{I}_{Z_e}(L + K_X))^*$  which is isomorphic, as we have already seen in (2.4), to the extension group  $\text{Ext}_{Z_e}^{n-1}(L) \square$

**Definition 2.3** *The rank of  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  is called the index of speciality (with respect to  $L$ ) of the universal cluster of  $\mathcal{E}$  over  $U_{\mathcal{E}}$  and it is denoted  $\delta(U_{\mathcal{E}})$  (or  $\delta(L, U_{\mathcal{E}})$  ).*

Arguing as in **Part I** of the introduction we can define a distinguished nonempty Zariski open subset  $U'_{\mathcal{E}}$  over which the sheaf  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1}$  is locally free of rank  $\delta(U_{\mathcal{E}})$ .

**Definition 2.4** Define

$$\mathbf{J}_{U_{\mathcal{E}}} := \mathbf{P}(\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1} \otimes \mathcal{O}_{U'_{\mathcal{E}}})$$

the projectivized bundle corresponding to  $\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1} \otimes \mathcal{O}_{U'_{\mathcal{E}}}$ . The variety  $\mathbf{J}_{U_{\mathcal{E}}}$  is called the non-abelian jacobian of the universal cluster of  $\mathcal{E}$  over  $U_{\mathcal{E}}$ .

The above terminology stems from considerations for surfaces, i.e.  $n = 2$ . In this case the variety  $\mathbf{J}_{U_{\mathcal{E}}}$  can be identified with the parameter space of a distinguished family of torsion free sheaves of rank 2 over  $X$  and its properties evoke a certain analogy with jacobians of smooth projective curves (see [R1] and §4, [R2]). For  $n \geq 3$  an interpretation of  $\mathbf{J}_{U_{\mathcal{E}}}$  as the parameter space of a family of torsion free sheaves of rank 3 is no longer valid. However we preserved the terminology for lack of a better one. It is clear that in dimension  $n \geq 3$  it is no longer bundles or torsion free sheaves but rather their complexes (up to a certain equivalence) which are parametrized by  $\mathbf{J}_{U_{\mathcal{E}}}$ . Thus the variety  $\mathbf{J}_{U_{\mathcal{E}}}$  can be regarded as the parameter space of a distinguished family of elements of the derived category of coherent sheaves on  $X$ . From this perspective the emergence of our jacobian fits naturally with recent understanding of the important role played by this category in geometry of projective varieties (see [B-O],[K]).

By definition  $\mathbf{J}_{U_{\mathcal{E}}}$  comes equipped with the natural projection

$$\pi_{U_{\mathcal{E}}} : \mathbf{J}_{U_{\mathcal{E}}} \longrightarrow U'_{\mathcal{E}} \quad (2.11)$$

and the line bundle  $\mathcal{O}_{\mathbf{J}_{U_{\mathcal{E}}}}(1)$  such that its direct image

$$\pi_{U_{\mathcal{E}}*} \mathcal{O}_{\mathbf{J}_{U_{\mathcal{E}}}}(1) = (\mathbf{Ext}_{U_{\mathcal{E}}}^{n-1})^*$$

As it was alluded to in the introduction the variety  $\mathbf{J}_{U_{\mathcal{E}}}$  should play a role of “hidden” parameters of the universal configuration  $\overset{\circ}{\mathcal{Z}}$  and it should reveal its “quantum” properties. It turns out that one can arrive to this by considering certain Hodge-like structure on the sheaf  $\pi_{U_{\mathcal{E}}}^*(p_{U_{\mathcal{E}}*}(\mathcal{O}_{\mathcal{Z}}))$ .

### § 3 Hodge-like structure on $\pi_{U_\varepsilon}^*(p_{U_\varepsilon*}(\mathcal{O}_Z))$

We begin by explaining our construction on a fibre of the projection  $\pi_{U_\varepsilon} : \mathbf{J}_{U_\varepsilon} \longrightarrow U'_\varepsilon$ .

Fix  $[e] \in U'_\varepsilon$ . On the fibre of  $\mathbf{J}_{U_\varepsilon}$  over  $[e]$  the sheaf  $\pi_{U_\varepsilon}^*(p_{U_\varepsilon*}(\mathcal{O}_Z))$  is just the trivial bundle  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$ . The Hodge-like structure we have in mind is a distinguished decreasing filtration on  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$ . In order to construct it we will need some preliminary observations.

**Lemma 3.1** *There is a natural morphism*

$$\theta_{[e]} : H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(-1) \longrightarrow (H^0(\mathcal{O}_{Z_e}(L + K_X)))^* \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} \quad (3.1)$$

which is an isomorphism at a general point  $[\alpha] \in \mathbf{P}(Ext_{Z_e}^{n-1}(L))$

*Proof.* By Serre duality

$$(H^0(\mathcal{O}_{Z_e}(L + K_X)))^* = Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X)) \quad (3.2)$$

The fact that  $Z_e$  is 0-dimensional implies that the sheaves  $Ext^i(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X)) = 0$ , for all  $i \leq n - 1$ . So the local-to-global spectral sequence for  $Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X))$  yields

$$Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X)) = H^0(Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X))) \quad (3.3)$$

The sheaf  $Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X))$  can be rewritten as follows

$$Ext^n(\mathcal{O}_{Z_e}(L + K_X), \mathcal{O}_X(K_X)) = Ext^n(\mathcal{O}_{Z_e}, \mathcal{O}_X(K_X)) \otimes \mathcal{O}_X(-L - K_X) = \omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X) \quad (3.4)$$

where  $\omega_{Z_e} = Ext^n(\mathcal{O}_{Z_e}, \mathcal{O}_X(K_X))$  is the dualizing sheaf of  $Z_e$ . This combined with (3.3) and (3.2) gives

$$(H^0(\mathcal{O}_{Z_e}(L + K_X)))^* = H^0(\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)) \quad (3.5)$$

Observe that  $H^0(\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X))$  is an  $H^0(\mathcal{O}_{Z_e})$ -module with the structure multiplication

$$H^0(\mathcal{O}_{Z_e}) \otimes H^0(\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)) \longrightarrow H^0(\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)) \quad (3.6)$$

From (2.9) we know that  $Ext_{Z_e}^{n-1}(L)$  is a subspace of  $(H^0(\mathcal{O}_{Z_e}(L + K_X)))^*$ . This combined with (3.5) and (3.6) yields the pairing

$$Y_e : H^0(\mathcal{O}_{Z_e}) \otimes Ext_{Z_e}^{n-1}(L) \longrightarrow H^0(\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)) = (H^0(\mathcal{O}_{Z_e}(L + K_X)))^* \quad (3.7)$$

which is equivalent to the sheaf morphism  $\theta_{[e]}$  asserted in the lemma. Turning to the second assertion of the lemma we recall the Koszul sequence (2.5) and the extension class  $\alpha_e$  defined by it. The pairing (3.7) with this class gives a homomorphism

$$Y_e(\cdot, \alpha_e) : H^0(\mathcal{O}_{Z_e}) \longrightarrow (H^0(\mathcal{O}_{Z_e}(L + K_X)))^*$$

which is known to be an isomorphism given explicitly by the residue map as in Chapter 5,[G-H].  $\square$

**Corollary 3.2** *The locus where  $\theta_{[e]}$  in (3.1) fails to be an isomorphism is a hypersurface  $\Theta([e])$  of degree  $d$  in  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$ . Furthermore for  $[e] \in \overset{\circ}{U}_{\mathcal{E}}$  and  $z \in Z_e$  the subset*

$$\Theta([e], z) = \{[\alpha] \in \mathbf{P}(Ext_{Z_e}^{n-1}(L)) \mid \alpha(z) = 0\} \quad (3.8)$$

*is a hyperplane in  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  and the support of  $\Theta([e])$  is equal to  $\bigcup_{z \in Z_e} \Theta([e], z)$ .*

*Proof.* Since the morphism  $\theta_{[e]}$  in (3.1) is generically an isomorphism its determinant  $\det \theta_{[e]}$  is a nonzero section of  $\mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(d)$  and the locus where  $\theta_{[e]}$  fails to be an isomorphism is the hypersurface

$$\Theta([e]) = (\det \theta_{[e]} = 0)$$

The second assertion of the corollary follows from the identification (3.5) and the fact that the dualizing sheaf  $\omega_{Z_e}$  of  $Z_e$  is an invertible sheaf on  $Z_e$ . This implies that we can identify  $Ext_{Z_e}^{n-1}(L)$  with the subspace of sections of an invertible sheaf on  $Z_e$ . This and the fact that there is a section in  $Ext_{Z_e}^{n-1}(L)$  which does not vanish anywhere on  $Z_e$  imply that  $\Theta([e], z)$  in (3.8) is a hyperplane in  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$ , for every  $z \in Z_e$ . It is then obvious that the union of those hyperplanes is contained in  $\Theta([e])$ . On the other hand it is clear that the complement of  $\bigcup_{z \in Z_e} \Theta([e], z)$  in  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  is contained in the complement of  $\Theta([e])$ . Hence the equality

$$\bigcup_{z \in Z_e} \Theta([e], z) = \text{support of } \Theta([e])$$

□

As  $[e]$  varies in  $U'_{\mathcal{E}}$  the hypersurfaces  $\Theta([e])$  will vary to form a divisor in  $\mathbf{J}_{U_{\mathcal{E}}}$  (see [R1], §1.2, for technical details). This divisor will be called the *theta-divisor* of  $\mathbf{J}_{U_{\mathcal{E}}}$  and denoted  $\Theta_{U_{\mathcal{E}}}$ .

**Definition 3.3** *An extension class  $\alpha \in Ext_{Z_e}^{n-1}(L)$  is called regular if the morphism  $\theta_{[e]}$  is an isomorphism at the point  $[\alpha] \in \mathbf{P}(Ext_{Z_e}^{n-1}(L))$  or, equivalently, if the homomorphism*

$$Y_{[e]}(\cdot, \alpha) : H^0(\mathcal{O}_{Z_e}) \longrightarrow (H^0(\mathcal{O}_{Z_e}(L + K_X)))^*$$

*induced by the pairing  $Y_{[e]}$  in (3.7) is an isomorphism.*

Once we have the morphism  $\theta_{[e]}$  in (3.1) we can define a distinguished subsheaf of  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$  (which will eventually become the first step of our filtration). This is done as follows. Tensor the morphism  $\theta_{[e]}$  with  $\mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(1)$  to obtain the morphism

$$\theta_{[e]}(1) : H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} \longrightarrow (H^0(\mathcal{O}_{Z_e}(L + K_X)))^* \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(1) \quad (3.9)$$

Define the subsheaf  $\tilde{\mathbf{H}}_{[e]}$  of  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$  to be the inverse image of the subsheaf  $Ext_{Z_e}^{n-1}(L) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(1)$  of  $(H^0(\mathcal{O}_{Z_e}(L + K_X)))^* \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(1)$  under the morphism  $\theta_{[e]}(1)$ . The following properties of the sheaf  $\tilde{\mathbf{H}}_{[e]}$  follow immediately from its definition and Corollary 3.2.

**Proposition 3.4** 1) The sheaf  $\tilde{\mathbf{H}}_{[e]}$  contains  $\mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$  as its subsheaf.

2) The morphism  $\theta_{[e]}(1)$  induces the following commutative diagram

$$\begin{array}{ccc}
 & 0 & \\
 & \downarrow & \\
 \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} & \xlongequal{\quad} & \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} \\
 & \downarrow & \downarrow \\
 \tilde{\mathbf{H}}_{[e]} & \xrightarrow{\theta_{[e]}(1)} & Ext_{Z_e}^{n-1}(L) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}(1) \\
 & \downarrow & \downarrow \\
 \tilde{\mathbf{H}}_{[e]}/\mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} & \xrightarrow{\theta'_{[e]}(1)} & \mathcal{T}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} \\
 & \downarrow & \downarrow \\
 & 0 & 
 \end{array} \tag{3.10}$$

where  $\mathcal{T}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$  is the tangent bundle of  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  and the column on the right is the Euler sequence on  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$ .

3) The morphism  $\theta_{[e]}(1)$  (resp.  $\theta'_{[e]}(1)$ ) in (3.10) is an isomorphism precisely on the complement of the divisor  $\Theta([e])$  in Corollary 3.2. In particular,  $\tilde{\mathbf{H}}_{[e]}$  is a torsion free sheaf on  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  and its singularity set is  $\Theta([e])$ . Furthermore, for general  $[e] \in U_{\mathcal{E}}$ , the rank of  $\tilde{\mathbf{H}}_{[e]}$  is equal to  $\delta(U_{\mathcal{E}})$ .

**Remark 3.5** If  $[\alpha]$  is a point of  $\mathbf{P}(Ext_{Z_e}^{n-1}(L))$  corresponding to a regular extension class  $\alpha$ , then from the definition of the morphism  $\theta_{[e]}$  in (3.1) it follows that the fibre  $\tilde{\mathbf{H}}_{[e]}([\alpha])$  of  $\tilde{\mathbf{H}}_{[e]}$  is the following subspace of functions on  $Z_e$

$$\tilde{\mathbf{H}}_{[e]}([\alpha]) = \left\{ \frac{\beta}{\alpha} \mid \beta \in Ext_{Z_e}^{n-1}(L) \right\} \tag{3.11}$$

where we think of elements of  $Ext_{Z_e}^{n-1}(L)$  as sections of the invertible sheaf  $\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)$  and the quotient of two sections of an invertible sheaf is a (rational) function. Furthermore the fact that  $\alpha$  is regular means that it is a section of  $\omega_{Z_e} \otimes \mathcal{O}_X(-L - K_X)$  which does not vanish anywhere on  $Z_e$ . Hence the quotient  $\frac{\beta}{\alpha}$  is a (regular) function on  $Z_e$ .

We are now in the position to define our filtration on  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))}$ .

1) Define

$$\tilde{\mathbf{H}}_{-1}([e]) := \tilde{\mathbf{H}}_{[e]} \tag{3.12}$$

to be the first step of filtration.

2) To define its next steps we use the multiplication in  $H^0(\mathcal{O}_{Z_e})$ . Namely, for  $k \geq 2$  consider the morphism

$$m_k([e]) : S^k \tilde{\mathbf{H}}_{[e]} \longrightarrow H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(Ext_{Z_e}^{n-1}(L))} \tag{3.13}$$

induced by the multiplication in  $H^0(\mathcal{O}_{Z_e})$ . Define

$$\tilde{\mathbf{H}}_{-k}([e]) := \text{im}(m_k([e])) \quad (3.14)$$

the image of  $m_k([e])$ .

The inclusion  $\tilde{\mathbf{H}}_{-k}([e]) \subset \tilde{\mathbf{H}}_{-k-1}([e])$  follows from the fact that the constant function 1 is a global section of  $\tilde{\mathbf{H}}_{-1}([e])$  (Proposition 3.4,1)). Thus we obtain a decreasing filtration of  $H^0(\mathcal{O}_{Z_e}) \otimes \mathcal{O}_{\mathbf{P}(\text{Ext}_{Z_e}^{n-1}(L))}$

$$\tilde{\mathbf{H}}_{-1}([e]) \subset \dots \subset \tilde{\mathbf{H}}_{-k}([e]) \subset \tilde{\mathbf{H}}_{-k-1}([e]) \subset \dots \quad (3.15)$$

**Remark 3.6** Let  $\tilde{\mathbf{H}}_{-\bullet}([e], [\alpha])$  be the fibre of the filtration (3.15) at a point  $[\alpha] \in \mathbf{P}(\text{Ext}_{Z_e}^{n-1}(L))$ . This filtration of  $H^0(\mathcal{O}_{Z_e})$  has a simple geometric interpretation. Namely, the subspace  $\tilde{\mathbf{H}}_{-1}([e], [\alpha])$  of functions on  $Z_e$  defines the obvious morphism

$$\tilde{\kappa}([e], [\alpha]) : Z_e \longrightarrow \left( \tilde{\mathbf{H}}_{-1}([e], [\alpha]) \right)^* \quad (3.16)$$

Then the subspace  $\tilde{\mathbf{H}}_{-k}([e], [\alpha])$  is the pullback of the homogeneous polynomial functions of degree  $k$  on  $\left( \tilde{\mathbf{H}}_{-1}([e], [\alpha]) \right)^*$ , i.e.

$$\tilde{\mathbf{H}}_{-k}([e], [\alpha]) = \tilde{\kappa}^*([e], [\alpha])(S^k \tilde{\mathbf{H}}_{-1}([e], [\alpha]))$$

In particular, the last piece of the filtration  $\tilde{\mathbf{H}}_{-\bullet}([e], [\alpha])$  is the subring  $\tilde{\kappa}^*(H^0(\mathcal{O}_{Z'_e}))$ , where  $Z'_e$  is the image of  $\tilde{\kappa}([e], [\alpha])$ .

We have described our filtration for a fixed  $[e]$ . As  $[e]$  varies in  $U'_\mathcal{E}$  the sheaves in (3.15) fit together to give the decreasing filtration

$$\tilde{\mathbf{H}}_{-1} \subset \dots \subset \tilde{\mathbf{H}}_{-k} \subset \tilde{\mathbf{H}}_{-k-1} \subset \dots \quad (3.17)$$

of the locally free sheaf  $\pi_{U_\mathcal{E}*}^*(p_{U_\mathcal{E}*}(\mathcal{O}_Z))$ . In particular, all sheaves in the filtration are torsion free since they are nonzero subsheaves of a locally free sheaf. Hence the ranks of the sheaves in (3.17) are well-defined.

**Definition 3.7** Define the function  $P_{U_\mathcal{E}}(k) = \text{rk } \tilde{\mathbf{H}}_{-k}$ , for every integer  $k \geq 1$ . This function is called the Hilbert function of the universal cluster of  $\mathcal{E}$  over  $U_\mathcal{E}$ . The smallest value  $w(U_\mathcal{E})$  such that

$$P_{U_\mathcal{E}}(k) = P_{U_\mathcal{E}}(w(U_\mathcal{E})), \forall k \geq w(U_\mathcal{E})$$

is called the weight of the universal cluster of  $\mathcal{E}$  over  $U_\mathcal{E}$ .

**Remark 3.8** 1) By definition and Proposition 3.4

$$P_{U_\mathcal{E}}(1) = \delta(U_\mathcal{E})$$

2)  $P_{U_\mathcal{E}}(w(U_\mathcal{E})) = \deg Z'_e$ , where  $Z'_e$  is the image of the morphism  $\tilde{\kappa}([e], [\alpha])$  in (3.16), for general  $([e], [\alpha]) \in \mathbf{J}_{U_\mathcal{E}}$ .

Let  $S$  be the union of the singularity sets of the sheaves  $\tilde{\mathbf{H}}_{-k}, k = 1, \dots, w(U_\varepsilon)$ , and let  $\mathbf{J}'_{U_\varepsilon}$  be the complement of  $S$  in  $\mathbf{J}_{U_\varepsilon}$ . This is the largest Zariski open subset of  $\mathbf{J}_{U_\varepsilon}$  where the filtration (3.17) is locally free, i.e. every sheaf in (3.17) is locally free over  $\mathbf{J}'_{U_\varepsilon}$ .

**Remark 3.9** Let  $U''_\varepsilon = \pi_{U_\varepsilon}(\mathbf{J}'_{U_\varepsilon})$ . This is a nonempty Zariski open subset of  $U_\varepsilon$ . Let  $\Theta''_{U_\varepsilon}$  be the part of the theta-divisor  $\Theta_{U_\varepsilon}$  lying over  $U''_\varepsilon$ , i.e.

$$\Theta''_{U_\varepsilon} = \Theta_{U_\varepsilon} \cap \pi_{U_\varepsilon}^{-1}(U''_\varepsilon)$$

Then we claim that

$$\mathbf{J}'_{U_\varepsilon} = \pi_{U_\varepsilon}^{-1}(U''_\varepsilon) \setminus \Theta''_{U_\varepsilon} \quad (3.18)$$

This follows from the fact that for every  $[e] \in U_\varepsilon$  the sheaves of the filtration (3.15) have constant ranks on the complement of the divisor  $\Theta([e])$  in Corollary 3.2. Indeed, let  $[\alpha]$  and  $[\alpha']$  be two distinct points of  $\mathbf{P}(Ext_{Z_e}^{n-1}(L)) \setminus \Theta([e])$ . Then from (3.11) it follows that the multiplication by the function  $f = \frac{\alpha}{\alpha'}$  induces an isomorphism

$$f : \tilde{\mathbf{H}}_{[e]}([\alpha]) \longrightarrow \tilde{\mathbf{H}}_{[e]}([\alpha'])$$

This and the definition of  $\tilde{\mathbf{H}}_{-k}([e])$  implies that the multiplication

$$f^k : \tilde{\mathbf{H}}_{-k}([e], [\alpha]) \longrightarrow \tilde{\mathbf{H}}_{-k}([e], [\alpha'])$$

is an isomorphism between the fibres of  $\tilde{\mathbf{H}}_{-k}$  at  $([e], [\alpha])$  and  $([e], [\alpha'])$ , for every  $k \geq 1$ .

## § 4 Theorem 0.2 and Theorem 0.3: mathematics and string theory

In this section we give proofs (or their sketch with reference to [R1] for details) of the results described in **Part II** of the introduction. This will be followed by the speculations about possible string theoretic interpretations of these results.

### § 4.1 Mathematics of Theorem 0.2

The direct sum decomposition (0.12) in Theorem 0.2 is a result of splitting of the filtration (3.17). To explain this we need to observe that over the locus  $\overset{\circ}{U}_\varepsilon$  parametrizing regular sections of  $\mathcal{E}$  the sheaf  $p_{\overset{\circ}{U}_\varepsilon}^* \mathcal{O}_{\overset{\circ}{Z}}$  acquires an extra-structure of being a sheaf of Frobenius algebras. Indeed, for  $[e] \in \overset{\circ}{U}_\varepsilon$ , the ring  $H^0(\mathcal{O}_{Z_e})$  comes equipped with the trace map

$$Tr_{[e]} : H^0(\mathcal{O}_{Z_e}) \longrightarrow \mathbf{C}$$

defined by the formula

$$Tr_{[e]}(f) = \sum_{z \in Z_e} f(z), \quad \forall f \in H^0(\mathcal{O}_{Z_e}) \quad (4.1)$$

The trace map gives rise to a non-degenerate symmetric bilinear pairing

$$q_{[e]} : H^0(\mathcal{O}_{Z_e}) \times H^0(\mathcal{O}_{Z_e}) \longrightarrow \mathbf{C} \quad (4.2)$$

defined as follows

$$q_{[e]}(f, g) = Tr_{[e]}(fg), \quad \forall f, g \in H^0(\mathcal{O}_{Z_e}) \quad (4.3)$$

If no confusion is likely we will not distinguish the above pairing and the quadratic form associated to it and refer to both as quadratic form.

The essential observation is as follows.

**Lemma 4.1** *There is a nonempty Zariski open subset  $\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}$  of  $\mathbf{J}'_{U_\varepsilon}$  lying over  $\overset{\circ}{U}_\varepsilon'' = \overset{\circ}{U}_\varepsilon \cap U_\varepsilon''$  such that for every  $([e], [\alpha]) \in \check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}$  the restriction of the quadratic form  $q_{[e]}$  to the subspaces  $\tilde{\mathbf{H}}_{-k}([e], [\alpha]), \forall k \geq 1$ , is again non-degenerate.*

For a proof of this result we refer to [R1], §2, Claim 2.2.

The quadratic forms  $q_{[e]}$  fit together to define the quadratic form  $\mathbf{q}$  on the sheaf  $\pi_{\overset{\circ}{U}_\varepsilon}^*(p_{\overset{\circ}{U}_\varepsilon*}\mathcal{O}_{\overset{\circ}{Z}})$ . Using this form we define

$$\mathbf{H}^p = \begin{cases} (\tilde{\mathbf{H}}_{-p})^\perp \cap \tilde{\mathbf{H}}_{-p-1}, & \text{if } p = 0, \dots, w(U_\varepsilon) - 1 \\ (\tilde{\mathbf{H}}_{-w(U_\varepsilon)})^\perp, & \text{if } p = w(U_\varepsilon) \end{cases} \quad (4.4)$$

where  $(\tilde{\mathbf{H}}_{-s})^\perp$  denotes the orthogonal complement of  $\tilde{\mathbf{H}}_{-s}$  in  $\pi_{\overset{\circ}{U}_\varepsilon}^*(p_{\overset{\circ}{U}_\varepsilon*}\mathcal{O}_{\overset{\circ}{Z}})$  with respect to the quadratic form  $\mathbf{q}$  and where we agree that  $\tilde{\mathbf{H}}_0 = 0$ .

From Lemma 4.1 it follows that over the open set  $\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}$  we have an orthogonal direct sum decomposition

$$\pi_{\overset{\circ}{U}_\varepsilon}^*(p_{\overset{\circ}{U}_\varepsilon*}\mathcal{O}_{\overset{\circ}{Z}}) \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}} = \bigoplus_{p=0}^{w(U_\varepsilon)} \mathbf{H}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}}^p \quad (4.5)$$

where  $\mathbf{H}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}}^p = \mathbf{H}^p \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}}$ . This gives us the first statement of Theorem 0.2.

From now on, unless said otherwise, we will be working over  $\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}$  and we omit the subscripts from the notation above, if no confusion is likely.

The summand  $\mathbf{H}^0$  plays a special role in our constructions. First of all, by definition (4.4), we have the following

$$\mathbf{H}^0 = \tilde{\mathbf{H}}_{-1} \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_\varepsilon}} \quad (4.6)$$

This identification implies the relative version of Proposition 3.4 which is nothing but properties (a) and (b) of Theorem 0.2,2).

**Definition 4.2** *Let  $[e] \in \overset{\circ}{U}_\varepsilon$ . A regular extension class  $\alpha \in Ext_{Z_e}^{n-1}(L)$  (see Definition 3.3) is called polarizing or a polarization of  $Z_e$  if the quadratic form  $q_{[e]}$  in (4.2) is non-degenerate on  $\tilde{\mathbf{H}}_{-k}([e], [\alpha]), \forall k \geq 1$ .*

Our terminology is of course borrowed from the usual Hodge decomposition of the cohomology of compact Kähler manifolds induced by a Kähler metric or a polarization on a manifold. In fact the decomposition (4.5) could be regarded as an analogue of Hodge decomposition. With this analogy in mind a polarizing extension class  $\alpha \in \text{Ext}_{Z_e}^{n-1}(L)$  could be envisaged as a kind of non-classical “Kähler” structure on  $Z_e$ . Thus the vertical (fibre) directions of the variety  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  could be viewed as the moduli space of non-classical “Kähler” structures, while in the horizontal directions one has the classical variation of configurations  $Z_e$  as  $[e]$  varies in  $\overset{\circ}{U}_{\mathcal{E}}$ .

## § 4.2 Mathematics of Theorem 0.3

The direct sum decomposition (4.5) can be viewed as a variation of the decomposition

$$H^0(\mathcal{O}_{Z_e}) = \bigoplus_{p=0}^{w(U_{\mathcal{E}})} \mathbf{H}^p([e], [\alpha]) \quad (4.7)$$

as a point  $([e], [\alpha])$  moves in  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  and where  $\mathbf{H}^p([e], [\alpha])$  is the fibre of  $\mathbf{H}_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}^p$  at  $([e], [\alpha])$ . The key observation is that the variation with respect to the parameter  $[\alpha]$  is entirely captured by the multiplication in the ring  $H^0(\mathcal{O}_{Z_e})$  by functions lying in the summand  $\mathbf{H}^0([e], [\alpha])$ . This follows from the identification (4.6) and the isomorphism  $\theta'$  in (0.14). So we will concentrate on the properties of the multiplication by functions in  $\mathbf{H}^0([e], [\alpha])$ . In particular, we want to understand how this multiplication acts on the direct sum decomposition (4.7).

**Lemma 4.3** *Let  $D(t) : H^0(\mathcal{O}_{Z_e}) \longrightarrow H^0(\mathcal{O}_{Z_e})$  be the homomorphism of multiplication by  $t \in \mathbf{H}^0([e], [\alpha])$ . Then the following holds.*

1) *The subspace*

$$\tilde{\mathbf{H}}_{-w(U_{\mathcal{E}})}([e], [\alpha]) = \bigoplus_{p=0}^{w(U_{\mathcal{E}})-1} \mathbf{H}^p([e], [\alpha]) \quad (4.8)$$

*is preserved by  $D(t)$ .*

2) *Let  $D_p(t)$  be the restriction of  $D(t)$  to the summand  $\mathbf{H}^p([e], [\alpha])$ . Then*

$$D_p(t) : \mathbf{H}^p([e], [\alpha]) \longrightarrow \mathbf{H}^{p-1}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha]) \quad (4.9)$$

*for every  $p = 0, \dots, w(U_{\mathcal{E}}) - 1$ .*

*Proof.* The first statement follows from the definition of  $w(U_{\mathcal{E}})$ . To prove the second we take  $h \in \mathbf{H}^p([e], [\alpha]) = (\tilde{\mathbf{H}}_{-p}([e], [\alpha]))^\perp \cap \tilde{\mathbf{H}}_{-p-1}([e], [\alpha])$ . Multiplying it by  $t \in \mathbf{H}^0([e], [\alpha])$  we obtain

$$D_p(t)(h) = th \in \tilde{\mathbf{H}}_{-p-2}([e], [\alpha]) = \tilde{\mathbf{H}}_{-p}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha]) \quad (4.10)$$

Furthermore since  $h$  is also in  $(\tilde{\mathbf{H}}_{-p}([e], [\alpha]))^\perp$  we have

$$q_{[e]}(th, h') = q_{[e]}(h, th') = 0, \quad \forall h' \in \tilde{\mathbf{H}}_{-p+1}([e], [\alpha])$$

This implies that  $th \in (\tilde{\mathbf{H}}_{-p+1}([e], [\alpha]))^\perp$ . Combining with (4.10) yields

$$\begin{aligned} D_p(t)(h) &= th \in (\tilde{\mathbf{H}}_{-p+1}([e], [\alpha]))^\perp \cap (\tilde{\mathbf{H}}_{-p}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha])) \\ &= \mathbf{H}^{p-1}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha]) \end{aligned}$$

□

Denote by  $D^-(t), D^0(t), D^+(t)$  the projections of  $D_p(t)$  on the first, second and the third summand of  $\mathbf{H}^{p-1}([e], [\alpha]) \oplus \mathbf{H}^p([e], [\alpha]) \oplus \mathbf{H}^{p+1}([e], [\alpha])$  respectively. Set

$$D^\pm(t) = \sum_{p=0}^{w-1} D_p^\pm(t) \text{ and } D^0(t) = \sum_{p=0}^{w-1} D_p^0(t) \quad (4.11)$$

Then the multiplication homomorphism  $D(t)$ , restricted to the subspace  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$ , admits the following decomposition

$$D(t) = D^-(t) + D^0(t) + D^+(t) \quad (4.12)$$

The components  $D^\pm(t)$  act on the decomposition in (4.8) as homomorphisms of degree  $\pm 1$ , i.e. they shift the grading in (4.8) by  $\pm 1$ , while  $D^0(t)$  is a homomorphism of degree 0, i.e. it preserves the decomposition in (4.8). Thus with every point  $([e], [\alpha]) \in \check{\mathbf{J}}_{U_\varepsilon}^\circ$  we can associate the subspace

$$\{D^0(t), D^\pm(t) \mid t \in \mathbf{H}^0([e], [\alpha])\} \quad (4.13)$$

of endomorphisms of  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$ . These endomorphisms could be viewed as rules for the infinitesimal variation of the direct sum decomposition in (4.8) along the directions of the fibre of  $\check{\mathbf{J}}_{U_\varepsilon}^\circ$  at  $([e], [\alpha])$ .

The family of the endomorphisms in (4.13) can be rewritten as homomorphisms

$$D^0([e], [\alpha]), D^\pm([e], [\alpha]) : \tilde{\mathbf{H}}_{-w}([e], [\alpha]) \longrightarrow (\mathbf{H}^0([e], [\alpha]))^* \otimes \tilde{\mathbf{H}}_{-w}([e], [\alpha]) \quad (4.14)$$

Finally, letting  $([e], [\alpha])$  vary in  $\check{\mathbf{J}}_{U_\varepsilon}^\circ$  the above homomorphisms fit together to yield the morphisms of sheaves

$$D^0, D^\pm : \tilde{\mathbf{H}}_{-w} \longrightarrow (\mathbf{H}^0)^* \otimes \tilde{\mathbf{H}}_{-w} \quad (4.15)$$

while the multiplication morphism

$$D : \tilde{\mathbf{H}}_{-w} \longrightarrow (\mathbf{H}^0)^* \otimes \tilde{\mathbf{H}}_{-w}$$

admits the following decomposition

$$D = D^- + D^0 + D^+ \quad (4.16)$$

The commutativity of multiplication now translates into the fact that  $D$  is a Higgs morphism of  $\tilde{\mathbf{H}}_{-w}$  with values in  $(\mathbf{H}^0)^*$  (see [S] or Definition 3.2,[R1]), i.e.

$$D^2 = D \wedge D = 0$$

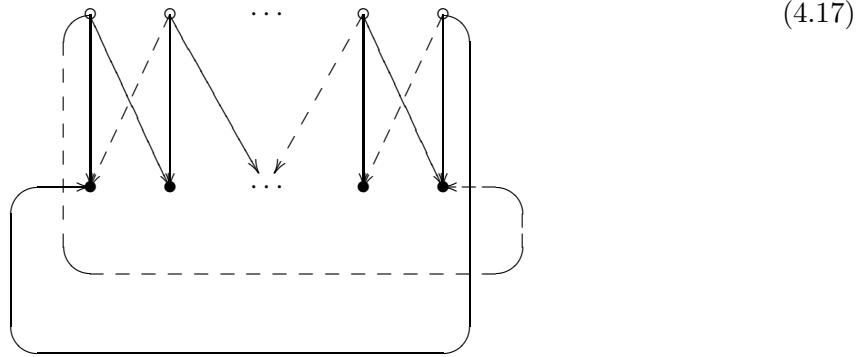
This together with the decomposition in (4.16) yields

$$0 = (D^- + D^0 + D^+) \wedge (D^- + D^0 + D^+)$$

Expanding and grouping together the terms of the same degree yield the relations (0.18) of Theorem 0.3.

### § 4.3 Trivalent graph associated to $\check{\mathbf{J}}_{\mathring{U}_\varepsilon}$

The data of the direct sum decomposition (0.15) and the morphisms  $D^0, D^\pm$  of Theorem 0.3 can be organized in the following trivalent graph



There are  $w = w(U_\varepsilon)$  pairs of vertices in (4.17) which are ordered by the index set  $I = \{0, \dots, w-1\}$  from left to right with the  $i$ -th pair being the ends of the  $i$ -th vertical edge in (4.17). These vertices are labeled by  $i_u$  and  $i_d$ , for the upper and lower one, respectively. Besides vertical edges relating  $i_u$  with  $i_d$ , for  $i \in I$ , the other edges are drawn as follows. For every  $i \in I$  the vertex  $i_u$  is connected to  $(i-1)_d$  and  $(i+1)_d$  with understanding that  $(-1)_d = (w-1)_d$  and  $w_d = 0_d$ . This graph will be denoted by  $G(\check{\mathbf{J}}_{\mathring{U}_\varepsilon})$  or simply by  $G$  if no ambiguity is likely.

We agree that all edges of  $G$  are oriented from “up” to “down”. The graph  $G$  with this orientation will be denoted by  $\vec{G}$ . In this orientation the edges incident to an upper (resp. lower) vertex of  $G$  are always out-going (resp. in-going). At every upper vertex of  $G$  we fix a counterclockwise order of incident edges and color them, starting with the vertical one, by “0”, “+”, and “−”, respectively. This will be called the *natural coloring of  $G$* .

It should be clear that the graph  $G$  with its coloring and orientation is just a pictorial representation of the decomposition (0.15) and the morphisms  $D^0, D^\pm$  in Theorem 0.3. Namely, if we place the summand  $\mathbf{H}^i$  of the direct sum decomposition at the vertices  $i_u$  and  $i_d$ , for every  $i \in I$ , and label the (oriented) edges at  $i_u$  by morphisms  $D^0, D^\pm$  so that the natural colors of edges are respected, i.e. we place  $D^0$  on the edge colored by “0” and  $D^\pm$  on the edges colored by “±” respectively, then we will recover the rules of action of the morphisms  $D^0, D^\pm$  on the summand  $\mathbf{H}^i$  placed at the vertex  $i_u$ . We will see shortly how one can use this pictorial representation to obtain “quantum” properties of the universal configuration.

### § 4.4 A string theoretic interpretation

Until now our construction of the variety  $\mathbf{J}_{U_\varepsilon}$  was motivated by purely algebro-geometric considerations. However, the higher *Ext* groups, the essential ingredient of the construction, have become familiar objects to physicists in connection with recent advances in understanding of the role and nature of  $D$ -branes in string theory (see [D] and the references therein). We will now attempt to review the variety  $\mathbf{J}_{U_\varepsilon}$  and its properties contained in Theorem 0.2 and Theorem 0.3 from the string theoretic perspective.

It is clear that the main idea of our construction is to consider 0-dimensional subschemes

$Z_e$  together with all nonzero extension classes  $\alpha \in \text{Ext}_{Z_e}^{n-1}(L)$ . We suggest to view a pair  $(Z_e, [\alpha])$  as a kind of  $D$ -brane structure on  $Z_e$ , i.e. thinking of  $\alpha$  as an equivalence class of complexes of sheaves on  $X$  (such as the Koszul complex (2.5), for example) one could view this class as “wrapped” on  $Z_e$ . The point is that though the individual sheaves of a complex representing  $\alpha$  will in general be globally supported on  $X$ , the equivalence class of a complex, the class  $\alpha$ , is locally supported on  $Z_e$  and it tells us something about  $Z_e$  and how it “seats” in  $X$ . With this point of view in mind, the fibre of  $\mathbf{J}_{U_{\mathcal{E}}}$  over  $U_{\mathcal{E}}$  can be thought of as a moduli space of  $D$ -brane structures on  $Z_e$ . Furthermore at the points corresponding to the regular extension classes (Definition 3.3) we have an explicit identification of the extension group  $\text{Ext}_{Z_e}^{n-1}(L)$  with the subspace  $\tilde{\mathbf{H}}_{[e]}([\alpha]) = \mathbf{H}^0([e], [\alpha])$  of functions on  $Z_e$  (see (3.11)). We propose to view this identification as a “linear representation” of strings having ends on the  $D$ -brane  $(Z_e, [\alpha])$ . Let us now explain this point.

We propose to view the direct sum decomposition

$$\tilde{\mathbf{H}}_{-w}([e], [\alpha]) = \bigoplus_{p=0}^{w-1} \mathbf{H}^p([e], [\alpha]) \quad (4.18)$$

which we have seen in (4.8), and the space of endomorphisms

$$\{D^0(t), D^{\pm}(t) \mid t \in \mathbf{H}^0([e], [\alpha])\} \quad (4.19)$$

in (4.13), as a “quantum” data associated to the  $D$ -brane structure  $(Z_e, [\alpha])$ . From this perspective the trivalent graph  $G$  in (4.17) becomes an explicit mechanism transforming “classical observables” =functions in  $\mathbf{H}^0([e], [\alpha])$ , into “quantum observables” =linear operators on  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$ . Namely, given a path  $\gamma$  of the graph  $G$  we define the path-operator

$$D_{\gamma}([e], [\alpha]) : (\mathbf{H}^0([e], [\alpha]))^{\otimes l(\gamma)} \longrightarrow \text{End}(\tilde{\mathbf{H}}_{-w}([e], [\alpha])) \quad (4.20)$$

where  $l(\gamma)$  is the length of  $\gamma$ , i.e. the number of edges composing  $\gamma$ . This is done as follows.

By definition  $\gamma$  is a sequence  $\{\gamma_1, \dots, \gamma_{l(\gamma)}\}$  of oriented edges of  $G$  such that the end of  $\gamma_i$  is the origin of  $\gamma_{i+1}$ , for every  $i = 1, \dots, l(\gamma) - 1$ . Define

$$D_{\gamma}([e], [\alpha])(t_1 \otimes \dots \otimes t_{l(\gamma)}) = D^{c(\gamma_{l(\gamma)})}(t_{l(\gamma)}) \circ \dots \circ D^{c(\gamma_1)}(t_1) \quad (4.21)$$

for all  $t_1, \dots, t_{l(\gamma)} \in \mathbf{H}^0([e], [\alpha])$ , where the right-hand side is the composition of the endomorphisms in (4.19) with  $c(\gamma_i) \in \{-, 0, +\}$  determined by the following rule:

if the orientation of the edge  $\gamma_i$  in  $\gamma$  coincides with the one in  $\tilde{G}$ , then  $c(\gamma_i)$  is the natural color of  $\gamma_i$ , otherwise  $c(\gamma_i)$  is the opposite of the natural color (with understanding that the opposite of the neutral color “0” is “0”). In particular, given  $t \in \mathbf{H}^0([e], [\alpha])$  we obtain the operator

$$D_{\gamma}([e], [\alpha])(t) = D_{\gamma}([e], [\alpha])(\underbrace{t \otimes \dots \otimes t}_{l(\gamma)-times}) = D^{c(\gamma_{l(\gamma)})}(t) \circ \dots \circ D^{c(\gamma_1)}(t)$$

i.e.  $t \in \mathbf{H}^0([e], [\alpha])$  is transformed into a composition of operators from the space (4.19). Acting with  $D_{\gamma}([e], [\alpha])(t)$  on  $\mathbf{H}^0([e], [\alpha])$  has an effect of propagation of elements of  $\mathbf{H}^0([e], [\alpha])$  through  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$  along the path  $\gamma$ .

Intuitively, the elements of  $\mathbf{H}^0([e], [\alpha])$  become string-like dynamic objects propagating

in space according to the pattern of the graph  $G$ . Since the above construction depends on  $(Z_e, [\alpha])$  one is led to think of these “strings” as attached to this particular  $D$ -brane structure. The identification

$$\theta_{[e]}(\alpha) : \mathbf{H}^0([e], [\alpha]) \longrightarrow \text{Ext}_{Z_e}^{n-1}(L)$$

itself can be viewed as a kind of “attaching” map of “strings” to the  $D$ -brane structure  $(Z_e, [\alpha])$ . Furthermore we can attach a particular “string” to each point  $(z, [\alpha]) \in (Z_e, [\alpha])$ . Indeed, take  $\delta_z$  to be the  $\delta$ -function supported at  $z$ , then its projection  $\delta_z^0$  into the summand  $\mathbf{H}^0([e], [\alpha])$  gives such a particular string corresponding to  $(z, [\alpha])$ . So one can say that  $D$ -brane structure recovers point-like properties of the configuration  $Z_e$ . But we obtain much more, since having  $\delta_z^0$ , we can form the family of path-operators  $D_\gamma([e], [\alpha])(\delta_z^0)$  indexed by the paths of the graph  $G$ . This now takes a form of “quantum” properties of the configuration  $Z_e$ : a point  $z \in Z_e$  represented by  $\delta_z^0$  is no longer confined to a particular location but via the action of the path-operators  $D_\gamma([e], [\alpha])(\delta_z^0)$  can make its appearance elsewhere in  $\tilde{\mathbf{H}}_{-w}([e], [\alpha])$ . In addition, the path-operators  $D_\gamma([e], [\alpha])(\delta_z^0)$  act on  $\delta_{z'}^0$ , for any  $z' \in Z_e$ , and this can be regarded as “quantum interaction” between distinct points (particles) in  $Z_e$ .

All these “strange” phenomena indicate that our variety  $\mathbf{J}_{U_\varepsilon}$  of “hidden” parameters indeed reveals “quantum” properties of the universal configuration  $\mathcal{Z}$ .

## § 5 Theorem 0.4 and Theorem 0.5

The results of Theorem 0.4 are obtained by reinterpreting the morphisms  $D, D^0, D^\pm$  in Theorem 0.3 in geometric terms, while Theorem 0.5 is an application of path-operators discussed in §4.4 in the geometric context of Theorem 0.4. In this section it always will be assumed that the weight  $w = w(U_\varepsilon) \geq 2$ .

### § 5.1 Theorem 0.4

The main idea is to take the morphism  $D$  in (0.16) and to perturb it according to the direct sum decomposition (0.15) in a such a way that the commutativity condition would be preserved. More precisely, set  $D_p^0$  (resp.  $D_p^\pm$ ) to be restrictions of the morphism  $D^0$  (resp.  $D^\pm$ ) to the  $p$ -th summand of the decomposition in (0.15). Then  $D$  admits the following decomposition

$$D = D^0 + D^+ + D^- = \sum_{p=0}^{w-1} D_p^0 + \sum_{p=0}^{w-2} D_p^+ + \sum_{p=0}^{w-2} D_{p+1}^- \quad (5.1)$$

We take now a sufficiently general deformation of  $D$  of the form

$$D(\mathbf{t}, \mathbf{x}, \mathbf{y}) = \sum_{p=0}^{w-1} t_p D_p^0 + \sum_{p=0}^{w-2} x_p D_p^+ + \sum_{p=0}^{w-2} y_p D_{p+1}^- \quad (5.2)$$

where  $\mathbf{t} = (t_p) \in \mathbf{C}^w$ ,  $\mathbf{x} = (x_p) \in \mathbf{C}^{w-1}$ ,  $\mathbf{y} = (y_p) \in \mathbf{C}^{w-1}$ , and we seek conditions on the parameters  $\mathbf{t}, \mathbf{x}, \mathbf{y}$  such that

$$D^2(\mathbf{t}, \mathbf{x}, \mathbf{y}) = D(\mathbf{t}, \mathbf{x}, \mathbf{y}) \wedge D(\mathbf{t}, \mathbf{x}, \mathbf{y}) = 0 \quad (5.3)$$

would hold, i.e.  $D(\mathbf{t}, \mathbf{x}, \mathbf{y})$  is a Higgs morphism of  $\tilde{\mathbf{H}}_{-w}$  with values in  $(\mathbf{H}^0)^*$ .

Using the relations (0.18) it is straightforward to deduce the following (see [R1], §4, for details).

**Lemma 5.1** *The set*

$$\hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}} = \{(z, \mathbf{x}, \mathbf{y}) \in \mathbf{C} \times \mathbf{C}^{w-1} \times \mathbf{C}^{w-1} \mid \mathbf{x} = (x_p), \mathbf{y} = (y_p), x_p y_p = z^2, p = 0, \dots, w-2\} \quad (5.4)$$

*is a set of solutions for (5.3), i.e. the morphism*

$$D(z, \mathbf{x}, \mathbf{y}) = z D^0 + \sum_{p=0}^{w-2} x_p D_p^+ + \sum_{p=0}^{w-2} y_p D_{p+1}^-$$

*satisfies (5.3), for all  $(z, \mathbf{x}, \mathbf{y}) \in \hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$ .*

The set  $\hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  is an affine variety. It admits an obvious  $\mathbf{C}^*$ -action, since scaling a solution of (5.3) by a constant is again a solution of (5.3).

**Definition 5.2** *The set*

$$H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*} = \hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}} / \mathbf{C}^*$$

*of homothety equivalent Higgs morphisms parametrized by  $\hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  is called the nonabelian Albanese of  $\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}$ .*

As usual, if no ambiguity is likely, we will omit the subscript in  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*}$ .

We will now give a projective description of the Albanese  $H$ .

**Proposition 5.3** *Let  $V$  be the vector space freely generated by symbols  $v_0, v_p^\pm, (p = 0, \dots, w-2)$  and let  $V^*$  be the dual of  $V$  with the dual basis  $T, X_p, Y_p, (p = 0, \dots, w-2)$ . Then  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*}$  is a subvariety of  $\mathbf{P}(V)$  defined by the quadratic equations*

$$X_p Y_p = T^2 \quad (p = 0, \dots, w-2)$$

*In particular,  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*}$  is a projective variety of dimension  $w-1$  and it has degree  $2^{w-1}$  in  $\mathbf{P}(V)$ . Furthermore its dualizing sheaf  $\omega_H = \mathcal{O}_H(-1)$ , i.e.  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*}$  is a Fano variety of dimension  $w-1$ .*

This projective description of  $H_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}^*}$  follows immediately from the description of  $\hat{H}_{\mathbf{j}_{\overset{\circ}{U}_{\mathcal{E}}}}$  in Lemma 5.1.

**Corollary 5.4** *Let  $w = w(U_{\mathcal{E}}) \geq 3$ . Then hyperplane sections of  $H$  are Calabi-Yau varieties of dimension  $w-2$ .*

*Proof.* The statement about the dualizing sheaf  $\omega_H$  of  $H$  in Proposition 5.3 and the adjunction formula (see e.g. [G-H]) yield the assertion.  $\square$

For other properties of  $H$ , in particular, its toric description we refer to §4,[R1].

## § 5.2 Theorem 0.5

This theorem establishes a geometric correspondence between the nonabelian jacobian  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  and its nonabelian Albanese  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$ . This correspondence is given by the morphism

$$CY_{\overset{\circ}{U}_{\mathcal{E}}} : \check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}} \longrightarrow \mathbf{P}(H^0(\mathcal{O}_{H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}})(d)) \quad (5.5)$$

which is called the Calabi-Yau cycle map for an obvious reason that the divisors assigned to points of  $\check{\mathbf{J}}$  are cycles of hyperplane sections of  $H$  which are, by Corollary 5.4, Calabi-Yau varieties. Before giving a construction of this map let us consider the situation from a more conceptual point of view.

We have constructed two varieties naturally associated to the universal configuration  $\overset{\circ}{Z}$ . The first one is  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  which as we have seen can be thought of as a moduli space of “classical” an “hidden” parameters of the universal configuration. The second one is the nonabelian Albanese  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$  which is a moduli of homothety equivalent Higgs structures on the sheaf  $\tilde{\mathbf{H}}_{-w} \otimes \mathcal{O}_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$ . It should be observed that  $H_{\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}}$  depends only on the weight  $w = w(U_{\mathcal{E}})$  and the relations (0.18) between morphisms  $D^0, D^{\pm}$ . In particular, it completely “forgets” about our variety  $X$  and the universal configuration  $\overset{\circ}{Z}$ . So a correspondence between  $\check{\mathbf{J}}_{\overset{\circ}{U}_{\mathcal{E}}}$  and its nonabelian Albanese  $H$  could be viewed as some kind of duality which partially restores ties of  $H$  with the variety  $X$ . In the discussion following the construction of  $CY_{\overset{\circ}{U}_{\mathcal{E}}}$  we will argue that this particular correspondence has a flavor of Mirror symmetry.

We now turn to the construction of the Calabi-Yau cycle map  $CY_{\overset{\circ}{U}_{\mathcal{E}}}$ . It is the same as in the proof of Proposition 5.1,[R1], but we reproduce it here as an illustration/application of the path-operator technique described in §4.4.

Let  $([e], [\alpha])$  be a point of  $\check{\mathbf{J}}$ . To define  $CY([e], [\alpha])$  we will produce a divisor of the form

$$\sum_{z \in Z_e} H_{z, [\alpha]}$$

where  $H_{z, [\alpha]}$  is a hyperplane section of  $H_{\check{\mathbf{J}}}$  intrinsically associated to  $(z, [\alpha]) \in (Z_e, [\alpha])$ . To do this we recall that  $H^0(\mathcal{O}_H(1))$  comes with a particular basis  $T, X_p, Y_p$  ( $p = 0, \dots, w-2$ ) (Proposition 5.3). So our strategy is as follows. For every point  $z \in Z$  we will produce the constants  $t(z, [\alpha]), x_p(z, [\alpha]), y_p(z, [\alpha])$  ( $p = 0, \dots, w-2$ ) depending holomorphically on  $(z, [\alpha])$ . Then we use these constants to define the section

$$s(z, [\alpha]) = t(z, [\alpha])T + \sum_{p=0}^{w-2} x_p(z, [\alpha])X_p + \sum_{p=0}^{w-2} y_p(z, [\alpha])Y_p \in H^0(\mathcal{O}_H(1)) \quad (5.6)$$

Then we define  $CY([e], [\alpha])$  to be the product  $\prod_{z \in Z_e} s(z, [\alpha])$ . Thus our argument comes down to defining the constants  $t(z, [\alpha]), x_p(z, [\alpha]), y_p(z, [\alpha])$  ( $p = 0, \dots, w-2$ ). This is done by using the orthogonal direct sum decomposition (0.15) at the point  $([e], [\alpha])$

$$\tilde{\mathbf{H}}_{-w}([e], [\alpha]) = \bigoplus_{p=0}^{w-1} \mathbf{H}^p([e], [\alpha]) \quad (5.7)$$

and the morphisms  $D^0, D^\pm$  in Theorem 0.3.

We proceed by attaching a particular “string” to the point  $(z, [\alpha])$  as discussed in §4.4, page 23. Namely, let  $\delta_z$  be the delta-function on  $Z$  supported at  $z$ , i.e.  $\delta_z(z) = 1$  and vanishes at other points of  $Z$ . Denote by  $\delta_z^0$  the component of  $\delta_z$  in  $\mathbf{H}^0([e], [\alpha])$ . Applying to it the operator  $D^+(\delta_z^0)$  we obtain (the right) moving string of functions  $\delta_z^{(p)} = (D^+(\delta_z^0))^p(\delta_z^0)$ , for  $p = 0, \dots, w-1$ , i.e. we move  $\delta_z^0$  along the shortest path of the graph  $G(\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ)$  in (4.17) connecting the first vertical edge of the graph with the last one and passing through every vertical edge in between.

Once we arrive to  $\delta_z^{(w-1)} \in \mathbf{H}^{w-1}([e], [\alpha])$  we apply the operator  $D^-(\delta_z^0)$  to create (the left) moving string of functions  $\delta_z^{(w-1), (w-1-m)} = (D^-(\delta_z^0))^m(\delta_z^{(w-1)})$ , for  $m = 0, \dots, w-1$  (here we move  $\delta_z^{(w-1)}$  back to the first vertical edge of the graph by retracing the shortest path used above). The desired constants are obtained essentially by evaluating all these functions at  $z$ . More precisely, define

$$x_p(z, [\alpha]) = \exp(\delta_z^{(p)}(z)), \quad y_p(z, [\alpha]) = \exp(\delta_z^{(w-1), (p+1)}(z)), \quad t(z, [\alpha]) = \exp(\delta_z^{(w-1), (0)}(z)),$$

where  $p = 0, \dots, w-2$ .

The above construction depends only on the orthogonal decomposition (5.7). Since the latter varies holomorphically with  $([e], [\alpha])$  we obtain that  $CY([e], [\alpha])$  depends holomorphically on  $([e], [\alpha])$  as well. This completes the proof of Theorem 0.5.

We want to argue that the geometric correspondence given by Calabi-Yau cycle map has a flavor of Mirror duality between nonabelian jacobian  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ$  and its nonabelian Albanese  $H_{\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ}$ .

Examining the construction of  $CY_{\dot{U}_\varepsilon}^\circ$  we see that the Calabi-Yau varieties appearing there come in families. Namely, for a fixed configuration  $Z_e$  the Calabi-Yau varieties  $H_{z, [\alpha]} (z \in Z_e)$  have moduli controlled by our “hidden” parameter  $[\alpha]$  varying in the fibre  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ([e])$  of  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ$

over  $[e] \in \dot{U}_\varepsilon^{''}$ . But we have already seen that this very parameter also controls non-classical “Kähler” structure of  $Z_e$  (see the discussion following Definition 4.2). Thus a parameter of non-classical “Kähler” structures on the side of  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ$  gets transformed into a classical parameter of variation of complex structure of hyperplane sections on the side of  $H_{\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ}$ . This can be

viewed as a half of Mirror duality between  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ$  and  $H_{\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ}$ . One would expect that the other half of Mirror duality holds as well, i.e. the Kähler moduli of the hyperplane sections of  $H_{\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ}$

gets transformed into the classical, “visible” parameter on the side of  $\check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ$ . In other words one should obtain interesting new invariants of the configurations  $Z_e$ , for  $[e] \in \dot{U}_\varepsilon^{''}$ , from the invariants of the families of Calabi-Yau varieties  $H_{z, [\alpha]} (z \in Z_e, [\alpha] \in \check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ([e]))$ . Furthermore

one can formulate the question of recovering the universal configuration  $\check{\mathcal{Z}}$  from the appropriate invariants of the family  $H_{z, [\alpha]} (z \in Z_e, [\alpha] \in \check{\mathbf{J}}_{\dot{U}_\varepsilon}^\circ([e]))$ . This gives an obvious analogue of the Torelli problem (see [G]).

Let us also observe that besides the continuous parameter  $[\alpha]$  the construction of  $CY_{\dot{U}_\varepsilon}^\circ$

contains a discrete parameter - the path of the graph  $G(\check{\mathbf{J}}_{U_\varepsilon}^\circ)$  chosen to create the constants defining the sections in (5.6). In the construction of  $CY_{U_\varepsilon}^\circ$  we have used the shortest (“geodesic”) path of the graph (the habits of classical mechanics are hard to die!). But of course we could choose any other suitable collection of paths. In fact, if we use the quantum mechanical principle of “summing over all histories”, we should sum over *all* suitable collections of paths of the graph  $G(\check{\mathbf{J}}_{U_\varepsilon}^\circ)$ . This will lead to a “quantum” version of the Calabi-Yau cycle map. These matters will be considered elsewhere (see §6, [R1], for related constructions of quantum-type invariants).

## § 6 Admissible families of clusters

In this section we generalize our constructions to families of clusters which do not necessarily come from vector bundles on  $X$ .

Examining our construction of variety  $\mathbf{J}_{U_\varepsilon}$  it becomes clear that the vector bundle has been used only to insure that every cluster of the family  $p_{U_\varepsilon} : \mathcal{Z} \longrightarrow U_\varepsilon$  is  $L$ -special. This can be formalized as follows.

Fix a line bundle  $\mathcal{O}_X(L)$  over a smooth complex projective variety  $X$  of dimension  $n \geq 2$ . We assume  $\mathcal{O}_X(L)$  to be subject to the vanishing condition in (2.1).

**Definition 6.1** 1) Let  $U$  be a smooth irreducible quasi-projective variety and let

$$p_U : \mathcal{Z}_U \longrightarrow U \tag{6.1}$$

be a family of clusters of degree  $d$  on  $X$ , i.e.  $p_U$  is finite morphism of degree  $d$  and the fibres of  $p_U$  are clusters of degree  $d$  on  $X$ . Such a family will be called a cluster of degree  $d$  on  $X$  over  $U$  or simply  $U$ -cluster of degree  $d$  on  $X$ .

2) A  $U$ -cluster of degree  $d$  on  $X$  is called generically smooth if the morphism  $p_U$  in (6.1) is generically smooth, i.e. for a general  $u \in U$ , the fibre  $Z_u$  is a set of  $d$  distinct points on  $X$ . Let  $\overset{\circ}{U}$  be the locus in  $U$  over which  $p_U$  is smooth. The cluster

$$\overset{\circ}{\mathcal{Z}}_U \longrightarrow \overset{\circ}{U}$$

is called  $U$ -configuration.

3) A  $U$ -cluster on  $X$  is called  $L$ -special if the fibre  $Z_u$  of  $p_U$  in (6.1) is  $L$ -special (Definition 2.1), for every  $u \in U$ .

It is clear that the construction in §2 extends to any  $L$ -special  $U$ -cluster of  $X$ .

**Proposition 6.2** Let  $p_U : \mathcal{Z}_U \longrightarrow U$  be an  $L$ -special  $U$ -cluster on  $X$ . There exists a torsion free sheaf  $\mathbf{Ext}_U^{n-1}$  over  $U$  such that for every  $u \in U$  its fibre  $\mathbf{Ext}_U^{n-1}(u)$  at  $u$  is isomorphic to  $\text{Ext}^{n-1}(\mathcal{I}_{Z_u}(L), \mathcal{O}_X)$ , where  $\mathcal{I}_{Z_u}$  is the ideal sheaf of the cluster  $Z_u$ .

**Definition 6.3** The rank of  $\mathbf{Ext}_U^{n-1}$  in Proposition 6.2 is called the index of  $L$ -speciality of a  $U$ -cluster and it will be denoted  $\delta(L, U)$ .

As in §2 we distinguish the Zariski open subset  $U'$  of  $U$  over which the sheaf  $\mathbf{Ext}_U^{n-1}$  has constant rank  $\delta(L, U)$  and define

$$\mathbf{J}_U(X; L, d) := \mathbf{P}(\mathbf{Ext}_U^{n-1} \otimes \mathcal{O}_{U'}) \quad (6.2)$$

where  $d$  is the degree of the  $U$ -cluster.

**Definition 6.4** *The variety  $\mathbf{J}_U(X; L, d)$  is called the nonabelian jacobian of an  $L$ -special  $U$ -cluster of degree  $d$  on  $X$ .*

If no ambiguity is likely we omit some or all of the parameters  $U, X, L, d$  in the above notation.

Let

$$\pi_U : \mathbf{J}_U \longrightarrow U' \quad (6.3)$$

be the natural projection and let  $\mathcal{O}_{\mathbf{J}_U}(1)$  be such that its direct image

$$\pi_{U*}\mathcal{O}_{\mathbf{J}_U}(1) = (\mathbf{Ext}_U^{n-1} \otimes \mathcal{O}_{U'})^*$$

We now turn to considerations of Hodge-like structure on the sheaf  $\pi_U^*(p_{U*}(\mathcal{O}_{Z_U}))$ . Again the only use of a vector bundle made in §3 was the existence of regular extension classes (Definition 3.3).

**Definition 6.5** *An  $L$ -special  $U$ -cluster on  $X$  is called regular if the extension group  $Ext^{n-1}(\mathcal{I}_{Z_u}(L), \mathcal{O}_X)$  contains a regular extension class, for every  $u \in U$ .*

This condition enables us to associate a distinguished sheaf-filtration

$$\tilde{\mathbf{H}}_{-1}(U) \subset \dots \subset \tilde{\mathbf{H}}_{-k}(U) \subset \tilde{\mathbf{H}}_{-k-1}(U) \subset \dots \quad (6.4)$$

of  $\pi_U^*(p_{U*}(\mathcal{O}_{Z_U}))$ . As before the filtration is torsion free and we can define the Hilbert function  $P_{L,U}$  and the weight  $w(L, U)$  of the  $U$ -cluster (with respect to the divisor  $L$ ) in exactly the same way as in §3, Definition 3.7.

To have the splitting of the filtration (6.4) and hence the generalization of Theorems in **Part II, III** of the introduction all we need is that our  $L$ -special, regular  $U$ -cluster would be generically smooth (Definition 6.1,2)).

**Definition 6.6** *An  $L$ -special, regular  $U$ -cluster on  $X$  is called  $L$ -admissible if it is generically smooth.*

**Corollary 6.7** *The Theorems 0.2 -0.5 in §0 hold for any  $L$ -admissible  $U$ -cluster on  $X$ .*

## § 7 Complete intersection clusters

We will illustrate our general contructions in the case of complete intersection clusters of sufficiently ample divisors.

Let  $X$  be a smooth complex projective variety of dimension  $n \geq 2$  (the case  $n = 2$  has been treated in [R1], §5.2). We will assume that the irregularity  $q(X) = h^1(X) = 0$ .

Fix a very ample line bundle  $\mathcal{O}_X(L)$  on  $X$  and consider 0-dimensional complete intersections defined by divisors of the linear system  $|L|$ . Let  $Z$  be such a cluster. Then identifying  $X$  with its image under the embedding defined by  $\mathcal{O}_X(L)$  we obtain  $Z$  by intersecting  $X$  with a projective subspace of codimension  $n$  or, dually,  $Z$  is defined by the global sections of  $\mathcal{O}_X(L)$  vanishing on  $Z$ , i.e. by the subspace  $H^0(\mathcal{I}_Z(L))$  of dimension  $n$  in  $H^0(\mathcal{O}_X(L))$ . Thus 0-dimensional complete intersections of divisors in  $|L|$  are parametrized by a Zariski open subset  $U = U(L)$  of the Grassmannian  $Gr(n, H^0(\mathcal{O}_X(L)))$  of  $n$ -planes in  $H^0(\mathcal{O}_X(L))$ . If  $W$  is an  $n$ -dimensional subspace of  $H^0(\mathcal{O}_X(L))$  we denote by  $[W]$  the corresponding point in  $Gr(n, H^0(\mathcal{O}_X(L)))$  and  $Z_W$  the complete intersection subscheme of  $X$  defined by  $W$ . In particular, it is a well-known fact in algebraic geometry (Bertini theorem, see e.g. [H]) that for a general  $[W] \in U$ , the complete intersection  $Z_W$  is smooth, i.e.  $Z_W$  is the set of  $L^n$  distinct points on  $X$ , where  $L^n$  is the self-intersection of  $L$  of degree  $n$ . Thus the  $U$ -cluster of complete intersections in  $|L|$  is generically smooth (Definition 6.1).

Next we consider the question of speciality with respect to the powers of  $\mathcal{O}_X(L)$ . Let  $[W] \in U$  and let  $Z_W$  be the corresponding complete intersection. The ideal sheaf  $\mathcal{I}_{Z_W}$  of  $Z_W$  admits the following Koszul resolution

$$\longrightarrow \wedge^2 W \otimes \mathcal{O}_X(-2L) \longrightarrow W \otimes \mathcal{O}_X(-L) \longrightarrow \mathcal{I}_{Z_W} \longrightarrow 0 \quad (7.1)$$

$$0 \longrightarrow \mathcal{O}_X(-nL) \longrightarrow \wedge^{n-1} W \otimes \mathcal{O}_X(-(n-1)L) \longrightarrow \cdots \longrightarrow$$

Let us examine the extension group  $Ext_{Z_W}^{n-1}(mL) = Ext^{n-1}(\mathcal{I}_{Z_W}(mL), \mathcal{O}_X)$ , for  $m > 0$ . By Serre duality it is isomorphic to  $H^1(\mathcal{I}_{Z_W}(mL + K_X))^*$ . The latter group can be computed from the spectral sequence of (7.1) tensored with  $\mathcal{O}_X(mL + K_X)$ . This yields that  $Ext_{Z_W}^{n-1}(mL)$  vanishes for all  $m > n$ . For  $m = n$  we obtain

$$Ext_{Z_W}^{n-1}(nL) = H^1(\mathcal{I}_{Z_W}(nL + K_X))^* = H^0(\mathcal{O}_X) \quad (7.2)$$

i.e. there exists a unique, up to a scalar multiple, nonzero extension class. This class is of course represented by the Koszul resolution (7.1). The first nontrivial case from the point of view of our construction occurs for  $m = n - 1$

$$Ext_{Z_W}^{n-1}((n-1)L) = H^1(\mathcal{I}_{Z_W}((n-1)L + K_X))^* = H^0(\mathcal{O}_X(L))/W = H^0(\mathcal{O}_X(L))/H^0(\mathcal{I}_{Z_W}(L)) \quad (7.3)$$

Furthermore this identification implies that any section of  $\mathcal{O}_X(L)$  whose zero-locus is disjoint from  $Z_W$  defines a regular extension class (Definition 3.3). Thus we have proved the following.

**Proposition 7.1** *The  $U$ -cluster of complete intersections in  $|L|$  is  $(n-1)L$ -admissible (Definition 6.6) with index of speciality*

$$\delta((n-1)L, U) = h^0(L) - n$$

In particular,  $\delta((n-1)L, U) \geq 2$ , unless  $(X, \mathcal{O}_X(L)) = (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ .

From now on it will be assumed that the index of speciality  $\delta((n-1)L, U) \geq 2$  and we will investigate the nature of the direct sum decomposition in Theorem 0.3. To do this we fix

$[W] \in U$  corresponding to a smooth complete intersection  $Z_W$  and a regular extension class  $\alpha \in \text{Ext}_{Z_W}^{n-1}((n-1)L)$ . From the identification (7.3) it follows that the filtration  $\tilde{\mathbf{H}}_{-\bullet}([W], [\alpha])$  is determined by geometry of the image of  $Z_W$  under the embedding defined by  $\mathcal{O}_X(L)$ . More precisely, the subspace  $\tilde{\mathbf{H}}_{-k}([W], [\alpha])$  is isomorphic to the image of the obvious homomorphism

$$S^k H^0(\mathcal{O}_X(L)) \longrightarrow H^0(\mathcal{O}_{Z_W}(kL)) \quad (7.4)$$

for every  $k \geq 1$ .

To go further with our computations we make the following assumptions on  $\mathcal{O}_X(L)$ :

- (i) the line bundle  $\mathcal{O}_X(L)$  is projectively normal, i.e. the homomorphism

$$S^k H^0(\mathcal{O}_X(L)) \longrightarrow H^0(\mathcal{O}_X(kL)) \quad (7.5)$$

is surjective, for every  $k \geq 1$ ;

(ii)

$$H^i(\mathcal{O}_X(kL)) = 0, \quad \forall k \geq 1 \text{ and } \forall i \geq 1 \quad (7.6)$$

Observe that both conditions are satisfied if  $L$  is a sufficiently high multiple of an arbitrary ample divisor  $D$  on  $X$ , i.e.  $L = tD$ , for an integer  $t >> 0$ .

The two assumptions above together with (7.4) imply

$$\dim \tilde{\mathbf{H}}_{-k}([W], [\alpha]) = \deg Z_W - h^1(\mathcal{I}_{Z_W}(kL)) = L^n - h^1(\mathcal{I}_{Z_W}(kL)) \quad (7.7)$$

To compute  $h^1(\mathcal{I}_{Z_W}(kL))$  we go back to the Koszul resolution (7.1) tensored with  $\mathcal{O}_X(kL)$ . The resulting spectral sequence together with (7.6) imply that  $H^1(\mathcal{I}_{Z_W}(kL)) = 0, \forall k \geq n+1$ . Thus the length of the filtration  $\tilde{\mathbf{H}}_{-\bullet}([W], [\alpha])$  or, equivalently, the weight  $w((n-1)L, U)$  of the  $U$ -cluster of complete intersections in  $|L|$  is at most  $n+1$ . Furthermore

$$H^1(\mathcal{I}_{Z_W}(nL)) = H^n(\mathcal{O}_X) = H^0(\mathcal{O}_X(K_X))^* \quad (7.8)$$

where the last equality is the Serre duality. Thus the filtration  $\tilde{\mathbf{H}}_{-\bullet}([W], [\alpha])$  has the following form

$$\tilde{\mathbf{H}}_{-1}([W], [\alpha]) \subset \tilde{\mathbf{H}}_{-2}([W], [\alpha]) \subset \cdots \subset \tilde{\mathbf{H}}_{-n}([W], [\alpha]) \subset \tilde{\mathbf{H}}_{-n-1}([W], [\alpha]) = H^0(\mathcal{O}_{Z_W}) \quad (7.9)$$

In particular, if  $\alpha$  is a polarizing class in  $\text{Ext}_{Z_W}^{n-1}((n-1)L)$ , then by Theorem 0.2 we have the following orthogonal direct sum decomposition

$$H^0(\mathcal{O}_{Z_W}) = \bigoplus_{p=0}^n \mathbf{H}^p([W], [\alpha]) \quad (7.10)$$

where  $\mathbf{H}^0([W], [\alpha]) \cong H^0(\mathcal{O}_X(L))/W$  and  $\mathbf{H}^n([W], [\alpha]) \cong H^0(\mathcal{O}_X(K_X))$ . Thus we proved the following.

**Proposition 7.2** *Let  $X$  be a smooth complex projective variety of dimension  $n \geq 2$  and  $q(X) = h^1(\mathcal{O}_X) = 0$ . Let  $\mathcal{O}_X(L)$  be a very ample line bundle on  $X$  subject to (7.5) and (7.6). Let  $U = U(L)$  be the variety parametrizing 0-dimensional complete intersections of divisors in the linear system  $|L|$ . Then the nonabelian jacobian  $\mathbf{J}_U(X; (n-1)L, L^n)$  is a  $\mathbf{P}^{\delta-1}$ -bundle*

over  $U$ , where  $\delta = \delta((n-1)L, U) = h^0(\mathcal{O}_X(L)) - n$ .

Furthermore, the weight with respect to  $(n-1)L$  of the  $U$ -cluster of complete intersections is  $w = w((n-1)L, U) \leq n+1$  and over the Zariski open subset  $\check{\mathbf{J}}_U(X; (n-1)L, L^n)$  of  $\mathbf{J}_U(X; (n-1)L, L^n)$  the sheaf  $\pi_U^*(p_{U*}(\mathcal{O}_{\mathcal{Z}_U}))$  has the following direct sum decomposition

$$\pi_U^*(p_{U*}(\mathcal{O}_{\mathcal{Z}_U})) \otimes \mathcal{O}_{\check{\mathbf{J}}_U} = \bigoplus_{p=0}^n \mathbf{H}^p$$

where the ranks of the first and the last summand are as follows

$$rk(\mathbf{H}^0) = \delta((n-1)L, U) = h^0(\mathcal{O}_X(L)) - n, \quad rk(\mathbf{H}^n) = p_g(X)$$

In particular, the weight  $w = w((n-1)L, U) = n+1$  if and only if  $p_g(X) > 0$ .

From this result and Theorem 0.4 we deduce the following.

**Corollary 7.3** *Let  $X, L$  and  $U$  be as in Proposition 7.2 and assume  $p_g(X) > 0$ . Then the nonabelian Albanese  $H$  of  $\check{\mathbf{J}}_U(X; (n-1)L, L^n)$  is a projective toric Fano variety whose hyperplane sections are Calabi-Yau varieties of dimension  $n-1$ .*

This and the construction of the Calabi-Yau cycle map in §5.2 imply that behind points of a smooth complex projective variety  $X$  of dimension  $n \geq 2$  with irregularity  $q(X) = 0$  and geometric genus  $p_g(X) > 0$  are “hidden” Calabi-Yau varieties of dimension  $n-1$ . To “reveal” them one needs to make a point of  $X$  to be a part of a smooth 0-dimensional complete intersection of  $n$  divisors in the linear system of sufficiently high multiple of an arbitrary ample divisor  $D$ . This could be seen as a mathematical evidence of the ubiquity of Calabi-Yau varieties which is certainly one of the predictions of string theory.

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